THE BOGOMOLOV-BEAUVILLE DECOMPOSITION

Sous la direction de
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DÉCLARATION D'INTÉGRITÉ RELATIVE AU PLAGIAT

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CONTENTS

INTRODUCTION

Le but de mon stage à l'Institut de Mathématiques de Marseille est de comprendre l'article [Bea83] de A. Beauville sur une décomposition pour les variétés kähleriennes dont la première classe de Chern s'annule. Ce théorème est la traduction pour les variétés complexes d'une théorème de décomposition pour les variétés riemanniennes et m'a amené naturellement aux trois tâches suivantes:

1. Comprendre la décomposition pour les variétés riemanniennes, qui est un résultat central de la théorie riemanienne. Ce théorème de decomposition était une contribution collective de générations de mathématiciens, de G. de Rham et E. Cartan à M. Berger. Mes références étaient [Joy00] et [Ber03] pour les grandes idées, puis [Sak96] pour les détails.

2. Comparer de différentes notions et notations développées dans la théorie riemannienne et la théorie complexe. Lorsque l'on travaille dans l'interface de ces deux théories, c'est une étape très essentielle et le livre [Huy05] y a consacré presque 2 chapitres (chapitre 1, puis l'appendice du chapitre 4).

3. La traduction (Théorème 28) n'étant pas formellement évidente, il me faut aussi m'habituer à la boite à outils qu'a utilisée A. Beauville. Cela contient essentiellement le théorème de Yau, le théorème de Bieberbach et la décomposition de Cheeger-Gromoll que j'ai appris dans [Bł12], [Bus85], [CG71] et [Bes07].

Ce mémoire se structure de manière suivante.

1. Le *théorème de de Rham*, présenté dans la section 1, permet de décomposer une variété riemannienne en somme (orthogonale) de variétés où l'action d'holonomie agit de manière irreductible sur chaque fibre tangente. On s'intéresse donc aux variétés irréductibles, qui sont les éléments constitutifs des variétés riemanniennes. On verra dans cette section que tout groupe/ toute representation n'est pas groupe/representation d'holonomie d'où la question que l'on pose naturellement: quels groupes/representations sont d'holonomie?

2. La théorie de groupes de Lie, développée par E. Cartan nous fournit une liste de groupe d'holonomie des *espaces symétriques*. Ce résultat sera présenté dans la section 2.

3. La *liste de Berger* dans la section 3 nous donne tous les possibilités d'un groupe d'holonomie d'une variété non-symétrique irréductible. Le théorème de Berger repond complètement à la question posée au-dessus, il nous permet aussi de voir que les variétés riemanniennes Ricci-plates sont, à un revêtement près, les produits d'un espace euclidien avec des variétés speciales unitaires et des variétés symplectiques irréductibles.

4. Dans la section 4, on verra la réponse de S.T. Yau pour une question posée par E. Calabi, dont une conséquence dans notre contexte est que *les variétés riemanniennes d'holonomie dans* $SU(m)$*, si on oublie leur métrique tout en gardant la structure (presque) complexe, sont exactement les variétés de type kählerien et de fibré canonique trivial (ou de première classe de Chern nulle si elle sont simplement connexes).* Le théorème de Yau nous permet de définir les variétés dite *de Calabi-Yau*.

5. La *décomposition de Cheeger et Gromoll* décrit un phénomène remarquable des variétes riemanniennes complètes de courbure de Ricci non-négative: la variété se décompose en produit le long d'une droite géodésique. Une conséquence de ce théorème fournit un détail technique (la compacité) dans le théorème de Bogomolov et Beauville. Le dernier sera présenté dans la section 6 où l'on mettra en jeu toutes les technologies introduites auparavant.

6. Le mémoire se conclut avec de nouveaux développements du résultat de Bogomolov et Beauville dans section 7. On expliquera aussi un nouveau résultat de Campana, Demailly et Peternell dans [CDP12] plus précisément un théorème de décomposition des variétés dont la courbure de Ricci est non-négative.

On note que toute variété dans ce mémoire est lisse et de dimension finie. Toute connection est de Levi-Civita et tout transport parallèle est par rapport à la connection de Levi-Civita.

Je souhaite presenter dans ce mémoire les idées aussi claires que possible, c'est la raison pour laquelle j'y inclus séparément une partie manuscrite où j'ai détaillé les preuves de quelques énoncés simples ou de nature calculatoire (les calculs de champs de Jacobi par exemple). Cette partie contient aussi mes solutions pour des exercices de [Sak96] pour la partie riemannienne et de [Huy05] pour la partie complexe.

1
DE RHAM DECOMPOSITION

1.1 Decomposition theorem of de Rham

We observe that if a manifold (M, g) is globally a product $(M_1, g_1) \times (M_2, g_2)$ then $Hol_g(M) = Hol_{g_1}(M_1) \times Hol_{g_2}(M_2)$ and the holonomy representation of M is reducible. A result of de Rham says that one can decompose a Riemannian manifold as product of ones with irreducible holonomy representation.

Theorem 1 (De Rham decomposition)**.** *Given (M, g) a simply-connected and complete Riemannian manifold, there exists a unique decomposition up to isometry and permutation of factors*

$$(M, g) = \prod_{i=1}^{n} (M_i, g_i)$$

where (M_i, g_i) are complete, simply connected Riemannian irreducible manifolds. Moreover the holonomy representation of M over T_xM is the product of holonomy representations of M_i over $T_{x_i}M_i$ where $x = (x_1, \dots, x_n)$

Sketch of proof. The proof of this theorem contains two steps:

1. Remark that if the holonomy group is reducible then locally M is a product of Riemannian manifolds, i.e. for every $x \in M$ there exists a neighborhood U containing x with $(U, g) = (M_1, g_1) \times (M_2, g_2)$.

2. Obtain the global product structure from local one. This is where completeness is used.

We now discuss the first point with a bit more details. Suppose that $T_xM = U_x \oplus V_x$ where U_x, V_x are stable under action of holonomy group, then by transporting U_x, V_x to the tangent space of any point y (as they are stable by holonomy, the result is independent of the curve along which the transport is taken), we obtain then two subbundles A and B of TM over M that are stable by parallel transport. Then for every vector field u_A in A and v in TM, $\nabla_v u_A \in A$. As the Levi-Civita connection is torsionless, one deduces $[u_A, v_A] = \nabla_{u_A} v_A - \nabla_{v_A} u_A$ remains in A. By Frobenius theorem, locally at a point $x \in M$, there exist two manifolds M_1, M_2 whose tangent spaces are A and B.

Theorem 2 (Frobenius)**.** *Given a distribution D that associates to each point x a k -dimensional hyperplane of T_xM such that:*

1. *D varies smoothly, i.e. for every x_0,there exist k smooth vector fields locally defined near x_0 such that at each point x close to x_0, they form a base of $D(x)$.*

2. *D is stable by Lie bracket, i.e. for every vector fields X, Y on M that take value in D, $[X, Y]$ takes value in D.*

Then at each point $x \in M$, there exists a maximal k -dimensional sub-manifold N of M containing x such that $D(y)$ is the tangent of N at y for every $y \in N$. The maximality means that every sub-manifold of M that satisfies this condition is an open sub-manifold of N.

For a complete proof that M is isometric to $M_1 \times M_2$, see [Sak96] (Lemma 6.8-Theorem 6.11, chapter III). □

1.2 Uniqueness

We note that the decomposition is unique in the following sense:

Proposition 2.1 (Uniqueness of de Rham decomposition)**.** *If M is decomposed as $p_1 : M \longrightarrow E \times \prod M_i$ and $p_2 : M \longrightarrow E' \times \prod M'_j$ where M_i, M'_j are irreducible and E, E' are maximal Euclidean components (i.e. none of M_i, M'_j are isometric to $\mathbb{R}$). Then up to a rearrangement of indice j the composed map $f = p_2 \circ p_1^{-1} : E \times \prod M_i \longrightarrow E' \times \prod M'_i$ are product of the isometries $f_E : E \longrightarrow E'$ and $f_i : M_i \longrightarrow M'_i$.*

We first explain the appearance of Euclidean components E, E'. They come from the parallel transport of trivial representations appeared in the decomposition on each fiber. We call them Euclidean because they are, up to an isometry, $\mathbb{R}^k$ with the usual metric. This follows from the fact that $\mathbb{R}$ with any Riemannian metric is isometric to $\mathbb{R}$ with the Euclidean metric.

We first note that the uniqueness stated in Proposition 2.1 comes from the uniqueness of the decomposition of each tangent fiber, we have the following lemma.

Lemma 3 (Uniqueness of fiber decomposition)**.** *Let $f : M \longrightarrow M'$ be an isometry that send $x \in M$ to $y \in M'$. Let*

$$T_x M = E \oplus^\perp \bigoplus_i^{\perp} V_i, \quad T_y M' = E' \oplus^\perp \bigoplus_j^{\perp} V'_j$$

be a decomposition of $T_x M$ and $T_y M'$ as direct sum of trivial subspaces E, E' and irreducible non-trivial subspaces V_i, V'_j under holonomy action. Then up to a rearrangement of j, the pushforward f_* send E to E' and V_i to V'_i.

Remark 1. *One may note that a similar result is not true for general representations: one can only prove the uniqueness of the irreducible factors up to isomorphism and their multiplicity. But the individual irreducible summands might not map to individual summands (however if one groups all irreducible summands of the same type, then each group maps to another).*

The supplementary property of holonomy representation put into use here is the following:

Remark 2. *The holonomy representation $H \subset SO(V)$ on a fiber $V = T_xM$ satisfies the property (H): if $V = V_1 \oplus^\perp V_2$ where V_i are stable by H then $H = H_1 \times H_2$ where $H_i := \{h \in H : h|_{V_j} = Id, j \neq i\}$. It is obvious that $H \supset H_1 \times H_2$, the other inclusion is a consequence of de Rham decomposition along V_1, V_2 and the fact that $Hol(M_1 \times M_2) = Hol(M_1) \times Hol(M_2)$.*

An example of representation that does not satisfies this property (H) is the group $G = \{\pm I_2\}$. Take $V_i = \mathbb{R}e_i$, then $G_1 = G_2 = \{I_2\}$ therefore $G_1 \times G_2 \neq G$. This also illustrates the fact that not every group (representation) is a holonomy group (representation).

We prove the following lemma, which implies Lemma 3.

Lemma 4 (Uniqueness of representation decomposition)**.** *Let $G \subset SO(V)$ be an orthonormal representation on a finite dimensional vector space V with property (H). Given any two orthogonal decompositions*

$$V = E \times \prod V_i = E' \times \prod V'_j$$

where G acts trivially on E, E', and V_i, V'_j are irreducible and of dimension larger than 2, one has $E = E'$ and $V_i = V'_i$ up to a rearrangement of index j.

Moreover, given $J \in Hom_G(V, V) \cap SO(V)$ then J sends E and V_i to themselves.

Proof. Note that since action of G is special orthonormal, any one dimensional subspace of V stable by G is trivial under G, that explains why we supposed V_i, V'_j are of dimension larger than 2. It suffices to see that every irreducible subspace N of V is either contained in E or equal to V_i.

Let pr_i and pr_E be orthogonal projection of V to V_i and E. As E and V_i are G -stable, these projections are G -invariant. Let $N_i = pr_i(N)$, then N_i is a subspace of V_i stable by G, hence either 0 or all V_i. If all $N_i = 0$ then clearly N is perpendicular to $\bigoplus_i V_i$, that is $N \subset E$. If $pr_E(N) \neq 0$ then $Hom_G(N, E) \neq 0$ since it contains pr_E. Since N is irreducible, N is G -isomorphic to an irreducible component of N by pr_E, therefore N is G -trivial hence $N \subset E$. Therefore one can suppose that at least one $N_i = V_i$ and $pr_E(N) = 0$, i.e. $E \perp N$. Note that pr_i is bijective by Schur lemma.

Let $G_i = \{g \in G : g|_{V_j} = Id \ \forall j \neq i\}$ then $Fix(\prod_{j\neq i} G_j) = E \oplus V_i$, in fact if $v = e + \sum v_i \in Fix(\prod_{j\neq i} G_j)$ where $e \in E, v_k \in V_k$, one has $g_j v_j = v_j \ \forall g_j \in G_j$, hence $g v_j = v_j \forall g \in G$, hence $v_j = 0$. Now note that as pr_i commutes with G and N_i is fixed by $\prod_{j\neq i} G_j$, N is also fixed by $\prod_{j\neq i} G_j$. Therefore $N \subset E \oplus^\perp V_i$, hence $N = N_i = V_i$ as $N \perp E$.

For the last point, note that as J commutes with all elements of G, J sends $Fix(\prod_{j\in A} G_j)$ to itself. Therefore $J|_E : E \longrightarrow E$ and $J|_{E\oplus V_i} : E \oplus V_i \longrightarrow E \oplus V_i$, hence by orthogonality J sends E and V_i to themselves. $\square$

1.3 Application of uniqueness lemma: decomposition for Kähler manifold

Now let apply the de Rham decomposition to a complete Riemannian manifold M with holonomy $U(n) \subset SO(2n)$ (called a *Kähler manifold*). There exists on a fixed fiber T_xM an automorphism J that preserves the Riemannian metric and satisfies $J^2 = -1$. By transporting J to every other fibers of TM one obtains a *almost complex structure* on M.

Applying Lemma 4 to J which obviously commutes with G an orthonormal representation, one can see that such structure J passes to every manifold M_i and the Euclidean component $\mathbb{R}^n$ and remains parallel on these manifolds. We proved that M is decomposed to $\mathbb{C}^{n/2} \times \prod M_i$ where M_i are Kähler manifold. The decomposition map is both a Riemannian isometry and a isomorphism between complex manifold (i.e. its preserves complex structure).

2 SYMMETRIC SPACES AND LIE GROUPS

2.1 Symmetric space

By de Rham decomposition, we now focus more on the building blocks: Riemannian manifolds with irreducible holonomy. The theory of Lie groups allows us to understand a block if it is *symmetric*.

Definition 1. *A Riemannian manifold M is called* <u>*symmetric*</u> *if for every $x \in M$, there exists an isometry s_x of M such that x is an isolated fixed point and $s_x^2 = Id$.*

Let $x \in M$ and $v \in T_xM$, we note by $\exp_x(v)$ the point of distance $|v|$ in the geodesic starting in x with velocity $v/|v|$. We remark that any isometry s_x with $s_x^2 = Id$ and x as isolated fixed point satisfies

$$s_x(\exp_x(v)) = \exp_x(-v). \tag{1}$$

In fact the eigenvalues of T_xs_x have to be 1 or -1, but as x is an isolated fixed point one has $T_xs_x = -Id$. Then s_x as an isometry sends the geodesic starting at x with velocity v to one starting at $s_x(x) = x$ with velocity $(s_x)_*v = -v$ and we have (1).

Equation (1) tells us that s_x is a reflection of center x on every geodesic passing by x. We can compose two reflections s_x, s_y to form a translation on the geodesic connecting x and y. This shows that a symmetric space is complete and the group of isometries of the form $s_x \circ s_y$ acts transitively on M.

Theorem 5 (Symmetric space)**.** *Let M be a symmetric Riemannian manifold then*

1. *M is complete.*
2. *Fix $x_0 \in M$, let G be the group generated by the isometries of form $s_x \circ s_y$, $x, y \in M$ and H is the subgroup containing elements of G that fix x_0. Then G is a Lie subgroup of the group $Isom(M)$ of isometries of M, G is connected by arc, H is a closed Lie subgroup of G and M is isometric to G/H. Moreover the holonomy group of M is H.*

Remark 3. *In general, for a Lie group G and a closed Lie subgroup H, if G has a metric left-invariant by G and right-invariant by H (i.e. the metric on $\mathfrak{g}$ is invariant by action of H by adjoint) then*

$$\mathfrak{g} = \mathfrak{h} \oplus^\perp \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

But if G/H is symmetric then one has the following extra information

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

It turns out that this condition is quite strong and allowed E. Cartan to classify all such pairs $(\mathfrak{g}, \mathfrak{h})$.

2.2 Locally symmetric space

The previous results can be extended to locally symmetric spaces.

Proposition-Definition 2. *Let M is a Riemannian manifold, the followings are equivalent*

1. *For every $x \in M$, there exists a neighborhood U of x and an isometry $s_x : U \longrightarrow U$ such that $s_x^2 = Id$ and x is the unique fixed point of s_x.*
2. *The curvature tensor R satisfies*

$$\nabla R = 0$$

If they are satisfied, M is called locally symmetric.

Theorem 6 (Locally symmetric space)**.** *Let M be a locally symmetric Riemannian manifold, then there exists a unique symmetric simply connected Riemannian manifold N such that M and N are locally isometric, i.e. for every $x \in M$ and $y \in N$, there exists neighborhoods U of x and V of y that are isometric.*

As a result, the reduced holonomy of M is the same as the holonomy of N.

2.3 Annex: Group of isometries as Lie group

We explain in this annex some subtle details: how can a group of isometries be a manifold. We state, with [MZ55] as reference, the following general result:

Theorem 7 (faithful + locally compact $\implies$ Lie)**.** *Let G be a group acting faithfully on a connected manifold M such that each action is C^1 and G is locally compact. Then G is a Lie group and the map $G \times M \longrightarrow M$ is C^1.*

Note that we equip a group of isometries with the **compact-open topology**, as M is locally compact and therefore second-countable (i.e. the topology admits a countable base), we see that a group of isometries is also second-countable (or completely separable). It suffices to prove the local compactness for the group of (all) isometries as this property is inherited by its closed subgroup. The details can be found in the book [KN63] of Kobayashi-Nomizu (Volume I, Theorem 4.7).

Theorem 8. *Let M be a connected, locally-compact metric space and G be the group of isometries of M, then*

1. *G is locally compact.*
2. *G_a the subset of isometries fixing a point $a \in M$ is compact.*
3. *If, in addition, M is compact then G is also compact.*

3
BERGER CLASSIFICATION AND REMARKS ON PARALLEL STRUCTURE

3.1 Our story so far

De Rham decomposition theorem allows us to split a Riemannian manifold under certain conditions (complete and connected) as Riemannian product of complete connected manifolds with *irreducible holonomy representation*. If an irreducible building block is *locally symmetric*, the theory of Lie groups developed by E. Cartan gave a complete list of holonomy of these spaces. We now shift our focus on non-symmetric irreducible manifolds.

3.2 Berger classification of non-symmetric irreducible manifolds

Theorem 9 (Berger classification)**.** *For a non-symmetric irreducible manifold, the holonomy representation has to be one of the following*

1. $SO(n)$,
2. $U(m) \subset SO(2m)$,
3. $SU(m) \subset SO(2m)$,
4. $Sp(r) \subset SO(4r)$,
5. $SO(r)Sp(1) \subset SO(4r)$,
6. $G_2 \subset SO(7)$,
7. $Spin(7) \subset SO(8)$.

where $n = 2m = 4r$ is the (real) dimension. The group $Sp(r)$ will be described later in section 3.5.

Here are some notations, note always that

$$Sp(m) \subset SU(2m) \subset U(2m) \subset SO(4m)$$

1. If $Hol(g) \subset U(m) \subset SO(2m)$, g is called a *Kähler metric*.
2. If $Hol(g) \subset SU(m) \subset SO(2m)$, g is called a *Calabi-Yau* metric. We will see that a Calabi-Yau metric is a Kähler metric that is also Ricci-flat.

3. If $Hol(g) \subset Sp(m) \subset SO(4m)$ then g is called a *hyperkähler* metric.

4. G_2 and Spin(7) are called *exceptional holonomies*

To sum up: hyperkähler $\longrightarrow$ Calabi-Yau $\longrightarrow$ Kähler.

But what do we mean by $U(n) \subset SO(2n)$? To embed $U(n)$ in $SO(2n)$ one needs to identify $\mathbb{C}$ and $\mathbb{R}^{2n}$, this can be done using an almost complex structure J of $\mathbb{R}^{2n}$. We will prove that when we change the almost complex structure, the embedded image of $U(n)$ in $SO(2n)$ always remains in the same conjugacy class, which corresponds to the fact that while holonomy representation is well-defined, the holonomy group in $SO(2n)$ is only defined up to its conjugacy class.

3.3 Almost complex structure

Definition 3. *A(n) (almost) complex structure J on a vector space V is an automorphism $J: V \longrightarrow V$ with $J^2 = -Id_V$. If V has a scalar product g, we suppose in addition that $g \circ J = J$.*

A(n) (almost) complex structure J on manifold M is a vector bundle automorphism $J : TM \longrightarrow TM$ that satisfies $J_x^2 = -Id_{T_xM}$ for every $x \in M$. If M is a Riemannian manifold, we assume in addition that $g \circ J = g$.

Let us first have a look at a complex structure J *on a fiber* (vector space) V. Here are some direct consequences:

Complexification. g and J extend in an unique way over $V_{\mathbb{C}}$, the complexification of V, to a Hermitian product $g_{\mathbb{C}}$ and a $\mathbb{C}$ -linear automorphism (also noted by J) and one still has $g_{\mathbb{C}} \circ J = g_{\mathbb{C}}$.

Eigenspaces. The complexified space $V_{\mathbb{C}}$ is decomposed to $V_{\mathbb{C}} = V^{1,0} \oplus^{\perp} V^{0,1}$ where $V^{1,0}$ and $V^{0,1}$ are eigenspaces (complex vector space) corresponding to eigenvalues i and $-i$ of J on $V_{\mathbb{C}}$. The orthogonality is by $g_{\mathbb{C}}$. The complex conjugate $\sum z_i x_i \mapsto \sum \bar{z}_i x_i$ where $z_i \in \mathbb{C}$ and $x_i \in V$ maps $V^{1,0}$ to $V^{0,1}$. Their dimensions are therefore the same.

Hermitian form. The fundamental form ω of (V, J) is defined by

$$\omega(a, b) = g(Ja, b) = -g(a, Jb) \quad \text{on } V$$

which is an antisymmetric real 2-form with $\omega \circ J = \omega$. V equipped with the following Hermitian form

$$h(a, b) = g(a, b) - i\omega(a, b) \quad \text{on } V$$

in the sense that $h(.,.)$ is $\mathbb{R}$ -bilinear with $h(Ja, b) = ih(a, b)$ and $h(a, Jb) = -ih(a, b)$.

Identification. One usually identifies (V, J) and $(V^{1,0}, i)$ as vector spaces equipped with complex structure, using the following map:

$$\iota_J : x \mapsto \frac{1}{2}(x - iJ(x))$$

which is $\mathbb{C}$ -linear in the sense of complex structure: $\iota_J(Jx) = i\iota_J(x)$. Note that $(V, -J)$ is also isomorphic to $(V^{0,1}, i)$ by the conjugate of ι_J: $x \mapsto \frac{1}{2}(x + iJ(x))$.

Now note that on we have on (V, J) an hermitian product $h(.,.)$ and on $(V^{1,0}, i)$ the restricted Hermitian product $g_{\mathbb{C}}$ of $V_{\mathbb{C}}$. The following lemma gives their relation (the proof is straightforward computation, see Manuscript).

Lemma 10. *The identification $(V, J) = (V^{1,0}, i)$ by ι_J gives*

$$\frac{1}{2}h = g_{\mathbb{C}}|_{V^{1,0}}$$

We can now embed $U(n)$ to $SO(2n)$, in other words $U(V^{1,0})$ to $SO(V)$ by the map $\phi \mapsto \tilde{\phi}$ as follow:
$$\begin{tikzcd} V \arrow[r, "\tilde{\phi}"] \arrow[d, "\iota_J"] & V \arrow[d, "\iota_J"] \\ V^{1,0} \arrow[r, "\phi"] & V^{1,0} \end{tikzcd}$$

Note that the correspondance $\phi \leftrightarrow \tilde{\phi}$ is one-to-one between $\{\phi : V^{1,0} \longrightarrow V^{1,0}\ \mathbb{R}\text{-linear}\}$ and $\{\tilde{\phi} : V \longrightarrow V\ \mathbb{R}\text{-linear}\}$. Then

1. ϕ is $\mathbb{C}$ -linear if and only if $\tilde{\phi} \circ J = J \circ \tilde{\phi}$.
2. ϕ preserves $g_{\mathbb{C}}$ if and only if $\tilde{\phi}$ preserves h. Taking the real and imaginary part, the latter is equivalent to the fact that $\tilde{\phi}$ preserves g and ω.
3. Every $\mathbb{C}$ -linear $\tilde{\phi}$ preserves orientation of $V^{1,0}$ as $\mathbb{R}^{2n}$ (note that the fact that $\tilde{\phi}$ preserves orientation or not is independent of how one identifies $V^{1,0}$ and $\mathbb{R}^{2n}$).

Hence for every J, $\phi \mapsto \tilde{\phi}$ gives a embedding of $U(V^{1,0})$ to $SO(V)$. An orthonormal base of $V^{1,0}$ and that of V give a embedding $U(n) \subset SO(2n)$.

Remark 4. *The image of $U(n)$ in $SO(2n)$ may depends on J and the orthonormal base of V, but its conjugacy class in $SO(2n)$ is uniquely defined. This is because every complex structure J is, up to a orthonormal conjugation,*

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

3.4 Complexified dual and forms, prelude to Kähler geometry

We state first some linear algebra facts, whose proofs are tedious and can be consulted in the Manuscript.

Lemma 11 (Linear algebra facts). *1. Let $V = W_1 \oplus W_2$ be R -module then the exterior algebra of V splits into*

$$\bigwedge^n V = \bigoplus_{p+q=n} \bigwedge^p W_1 \otimes \bigwedge^q W_2$$

Note that the tensor product here is formal, and not related to the tensor product defining the exterior algebra.

2. If V has a complex structure J then J gives a complex structure on $V^* = Hom_{\mathbb{R}}(V, \mathbb{R})$, naturally by $\phi \mapsto \phi \circ J$.

One has

$$(V^*)_{\mathbb{C}} = Hom_{\mathbb{R}}(V, \mathbb{C}) \equiv Hom_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C})$$

and

$$(V^*)^{1,0} = Hom_{\mathbb{C}}((V, J), \mathbb{C}), \quad (V^*)^{0,1} = Hom_{\mathbb{C}}((V, -J), \mathbb{C})$$

where $Hom_{\mathbb{C}}(V, \mathbb{C})$ denotes the set of $\mathbb{R}$ -linear morphisms that preserves complex structures ($\mathbb{C}$ is implicitly with the complex structure $z \mapsto iz$)

Therefore $(V^*)_{\mathbb{C}} = (V^*)^{1,0} \oplus (V^*)^{0,1}$ is rewriten as

$$Hom_{\mathbb{R}}(V, \mathbb{C}) = Hom_{\mathbb{C}}((V, J), \mathbb{C}) \oplus Hom_{\mathbb{C}}((V, -J), \mathbb{C})$$

Using the first point of Lemma 11, one has

$$\bigwedge^n (V^*)_{\mathbb{C}} = \bigoplus_{p+q=n} \bigwedge^{p,q} (V^*)_{\mathbb{C}}$$

where $\bigwedge^{p,q} T^*_{\mathbb{C}} M$ denotes the $\mathbb{C}$ -vector space of forms that are p times $\mathbb{C}$ -linear and q times $\mathbb{C}$ -antilinear.

Note one can easily find in V an orthonormal basis $\partial_{x_i}, \partial_{y_i}$ with $J(\partial_{x_i}) = \partial_{y_i}$. We clarify here the definition and implicit identifications of basic objects such as dz_i and $d\bar{z}_i$.

$$\begin{tikzcd}
(V,J) \arrow[r, "dz_i"] \arrow[d, "\iota_J"'] & \mathbb{C} \\
(V^{1,0}, i) \arrow[ur, "dz_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "dz_i"] \arrow[d, "\mathbb{C}\text{-lin}"'] & \mathbb{C} \\
V_{\mathbb{C}} \arrow[ur, "dz_i"'] &
\end{tikzcd}$$

Figure 1: Two natural ways to define dz_i on $V^{1,0}$. They gives the same form.

$$\begin{tikzcd}
(V,-J) \arrow[r, "d\bar{z}_i"] \arrow[d, "\iota_J"'] & \mathbb{C} \\
(V^{0,1}, i) \arrow[ur, "d\bar{z}_i"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "d\bar{z}_i"] \arrow[d, "\mathbb{C}\text{-lin}"'] & \mathbb{C} \\
V_{\mathbb{C}} \arrow[ur, "d\bar{z}_i"'] &
\end{tikzcd}$$

Figure 2: Two natural ways to define $d\bar{z}_i$ on $V^{0,1}$. They gives the same form.

Object	**Where it belongs/ properties**	**Extension/ properties**
$\partial_{z_i} = \iota_J(\partial_{x_i})$ $= \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$	$V^{1,0}$, form a $\mathbb{C}$ -base	$dz_i(\partial_{z_j}) = \delta_{i,j}$, $dz_i(\partial_{\bar{z}_j}) = 0$
$\partial_{\bar{z}_i} = \iota_{-J}(\partial_{x_i})$ $= \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$	$V^{0,1}$, form a $\mathbb{C}$ -base	$d\bar{z}_i(\partial_{z_j}) = 0$, $d\bar{z}_i(\partial_{\bar{z}_j}) = \delta_{i,j}$
$dz_i = dx_i + idy_i$	$Hom_{\mathbb{C}}((V,J),\mathbb{C}) \equiv Hom_{\mathbb{C}}(V^{1,0},\mathbb{C})$, $\mathbb{C}$ -linear	$Hom_{\mathbb{C}}(V_{\mathbb{C}},\mathbb{C})$, null on $V^{0,1}$
$d\bar{z}_i = dx_i - idy_i$	$Hom_{\mathbb{C}}((V,-J),\mathbb{C}) \equiv Hom_{\mathbb{C}}(V^{0,1},\mathbb{C})$, $\mathbb{C}$ -antilinear	$Hom_{\mathbb{C}}(V_{\mathbb{C}},\mathbb{C})$, null on $V^{1,0}$

Remark 5. *One can note that there are two natural ways to extend dz_i to $V^{1,0}$*

1. *by first make a $\mathbb{C}$ -linear extension on $V_{\mathbb{C}}$, then make a restriction on $V^{1,0}$*
2. *using the identification $(V,J) \equiv (V^{1,0}, i)$*

but these two coincide, as there exists a unique form $\mathbb{C}$ -linear dz_i that satisfies $dz_i(\partial_{z_j}) = \delta_{i,j}, dz_i(\partial_{\bar{z}_j}) = 0$. Same story with $d\bar{z}_i$. See Figure 3.4 and Figure 3.4.

Proposition-Definition 4. *The following properties are equivalent and X is called a Kähler manifold if one of them is satisfied.*

1. *X is a complex manifold, equipped with a Hermitian structure $h(.,.)$ compatible with the complex structure J, and a fundamental form ω with $d\omega = 0$.*

2. *X is a Riemannian manifold with a parallel complex structure.*

3. *X is a complex manifold, equipped with a Hermitian structure such that the Chern connection on $T^{1,0}X$ is, up to an identification by ι_J, the Levi-Civita connection.*

4. *X is a complex manifold, equipped with an Hermitian structure such that the Chern connection on $T^{1,0}X$ is torsionless.*

We call a complex manifold X of Kähler type if there exists a Hermitian structure under which X is Kähler.

The proof is straightforward. The only part that is not trivial is that a parallel almost complex structures has to come from a complex atlas, i.e. atlas of X such that each transition map preserves the complex structure. Such almost complex structures are called *integrable*.

To prove this, one uses the following (1,2)-tensor called *Nijenhuis tensor* of a (1,1)-tensor A, defined by:

$$N_A(X,Y) = -A^2[X,Y] + A[AX,Y] + A[X,AY] - [AX,AY]$$

and the following theorem.

Theorem 12 (Newlander–Nirenberg). *An almost complex structure on M with vanishing N_J is integrable.*

The proof that a parallel almost complex structure J has $N_J = 0$ is computational in nature and can be found in the Manuscript.

3.5 Symplectic holonomy

One can look at the symplectic group $Sp(r)$ from the following two points of view:

1. $Sp(r)$ is the quaternionic unitary group, i.e. the subgroup of $Aut_{\mathbb{H}}(\mathbb{H}^r)$ of elements preserving a quaternion Hermitian form q, where $\mathbb{H}$ is the algebra of quaternions.

2. $Sp(r) = U(2r) \cap Sp(2r, \mathbb{C})$.

The second point of view explains how $Sp(r)$ is embedded in $SO(4r)$. Let us consider $Sp(r)$ from the first point. In our context, let V be a tangent space at one point of the manifold M, that is a $4r$ real dimensional vector space, one can regard $Sp(r)$ as the group of automorphisms of V preserving the Riemannian metric $g|_V$ and the complex structures I, J (hence $K = IJ$) satisfying $IJ = -IJ$. Hence we have the following remark:

Remark 6. *The following properties are equivalent for a Riemannian manifold M:*

1. *$Hol(M) \subset Sp(r) \subset SO(2r)$.*

2. *There exists on M three parallel complex structures I, J, K that satisfy $K = IJ = -JI$.*

3. *There exists on M a parallel complex structure I and a holomorphic (w.r.t I), parallel, 2-form φ that is non-degenerate at a point (hence at every point).*

We note that the holomorphic 2-form in the third point is given by

$$\varphi = \omega_J + \sqrt{-1}\omega_K$$

where ω_I and ω_K are fundamental forms with respect to complex structures I and K, and M is regarded under the complex structure I.

The implication (2) $\implies$ (3) is actually [Huy05, Exercise 1.2.5].

For the implication (3) $\implies$ (2), note that the real and imaginary part of φ are parallel, they correspond to complex structures J and K on M. Since φ is a (2,0)-form w.r.t I, one has $\varphi(Iu, v) = i\varphi(u, v)$, taking the real part and using the fact that g is non-degenerate, one has $K = IJ = -JI$.

4
CALABI-YAU THEOREM

4.1 Calabi conjecture

In complex geometry, one usually defines the *Ricci curvature* to be the real (1,1)-form ρ with $\rho(u,v) = Ric(Ju,v) = \mathrm{tr}(w \mapsto R(w,v).Ju)$, where R is the Riemannian curvature tensor, as ρ has the advantage of being an antisymmetric form. We will call ρ the Ricci form.

We start with the following fact, which is [Huy05, Exercise 4.A.3].

Proposition 12.1 (Ricci curvature and first Chern class). *Let (X,g) be a compact Kähler manifold. Then $i\rho(X,g)$ is the curvature of the Chern connection on the canonical bundle K_X. In other words, $\rho(X,g) \in -2\pi c_1(K_X)$ where $c_1(K_X)$ is the first Chern class of K_X.*

Remark 7. *For our convenience when talking about positivity, we would rather use the anticanonical bundle. Then K_X^{-1} is positive (resp. semi-positive) if and only if Ric is positive definite (resp. positive semi-definite) as a symmetric form.*

Definition 5. *The quadruple (h, g, ω, J) is said to be compatible if $g \circ J = g$ and $\omega(a,b) = g(Ja,b)$ and $h = g - i\omega$.*

Remark 8.
1. *When J is fixed, one of h, g, ω that is invariant by J determines the two others.*
2. *For a compatible quadruple, the condition $\nabla J = 0$ is equivalent to $d\omega = 0$. The fundamental form ω that satisfies $d\omega = 0$ is called a Kähler form.*

The Calabi conjecture asked whether for each form $R \in c_1(K_X)$ one can find a metric g' whose new fundamental form ω' is in the same Kähler class and $Ric(X, g') = R$. We prefer to work with the fundamental form instead of the metric g as the former is antisymmetric and its derivative is hence easy to define.

4.2 Reduction to local charts, Calabi-Yau theorem

h, g, ω in local coordinates. We note by $h_{i\bar{j}} = h(\partial_{x_i}, \partial_{x_j}) = 2g_{\mathbb{C}}(\partial_{z_i}, \partial_{z_j})$. By straightforward calculation one has

$$\begin{aligned}\omega &= -\frac{1}{2}\mathrm{Im}\ h_{i\bar{j}}(dx^i \wedge dx^j + dy^i \wedge dy^j) + \mathrm{Re}\ h_{i\bar{j}} dx^i \wedge dy^j \\ &= \frac{i}{2} h_{i\bar{j}} dz^i \wedge d\bar{z^j}\end{aligned}$$

and the condition $d\omega = 0$ is equivalent to

$$\frac{\partial h_{i\bar{j}}}{\partial z_k} = \frac{\partial h_{k\bar{j}}}{\partial z_i}.$$

We also note by $h^{i\bar{j}}$ the inverse transposed of $h_{i\bar{j}}$, i.e. $h^{i\bar{j}} h_{k\bar{j}} = \delta_j^k$.

Definition 6. *Let X be an almost complex manifold (manifold with an almost complex structure). Then $d : \bigwedge^n T^*X \longrightarrow \bigwedge^{n+1} T^*X$ sends $\bigwedge^{p,q} T^*M$ to $\bigwedge^{p+1,q} T^*M \oplus \bigwedge^{p,q+1} T^*M$. We denote by ∂ and $\bar{\partial}$ the component of d in $\bigwedge^{p+1,q} T^*M$ and $\bigwedge^{p,q+1} T^*M$ respectively.*

It would be convenient to define $d^c = i(\bar{\partial} - \partial)$ then obviously $dd^c = 2i\partial\bar{\partial}$.

The Ricci curvature. The Ricci curvature form is given in local coordinates by

$$Ric_\omega = -\frac{1}{2} dd^c \log\det(h_{i\bar{j}}).$$

dd^c lemma . We then can state the dd^c lemma.

Lemma 13. *Let α be a real, (1,1)-form on a compact Kähler manifold M. Then α is d -exact if and only if there exists $\eta \in C^\infty(M)$ globally defined such that $\alpha = dd^c\eta$.*

Yau's theorem. The dd^c lemma tells us that every form $R \in c_1(K_X)$ is of form $Ric_\omega + dd^c\eta$. If one varies the Hermitian product $h_{i\bar{j}}$ to $h_{i\bar{j}} + \phi_{i\bar{j}}$ then the new Ricci curvature is $dd^c \log\det(h_{i\bar{j}} + \phi_{i\bar{j}})$. The Calabi conjecture can be restated as the existence of ϕ such that $h_{i\bar{j}} + \phi_{i\bar{j}}$ is definite positive and

$$dd^c \Big(\log\det(h_{i\bar{j}} + \phi_{i\bar{j}}) - \log\det(h_{i\bar{j}}) - \eta\Big) = 0 \tag{2}$$

The functions f that satisfies $dd^c f = 0$ are called *pluriharmonic*. They also satisfy the maximum principle. By compactness of X, these functions on X are exactly constant functions. Therefore (2) is equivalent to

$$\det(h_{i\bar{j}} + \phi_{i\bar{j}}) = e^{c+\eta} \det(h_{i\bar{j}})$$

or by dd^c lemma:

$$(\omega + dd^c\phi)^n = e^{c+\eta}\omega^n$$

where ω^n denotes the repeated wedge product. Note that $(\omega + dd^c\phi)^n - \omega^n$ is exact, one has $\int_M (\omega + dd^c\phi)^n = V$, the conjecture of Calabi is therefore a consequence of the following theorem.

Theorem 14 (Yau)**.** *Given a function $f \in C^\infty(M), f > 0$ such that $\int_M f\omega^n = V$. There exists, uniquely up to constant, $\phi \in C^\infty(M)$ such that $\omega + dd^c\phi > 0$ and*

$$(\omega + dd^c\phi)^n = f\omega^n. \tag{3}$$

4.3 A sketch of proof

The uniqueness is straightforward. In fact if ϕ and ψ both satisfy $\omega + dd^c\phi > 0$, $\omega + dd^c\psi > 0$ and $(\omega + dd^c\phi)^n = (\omega + dd^c\psi)^n$ then $D(\phi - \psi) = 0$ as

$$0 = \int_M (\phi - \psi)((\omega + dd^c\phi)^n - (\omega + dd^c\psi)^n) = \int_M d(\phi - \psi) \wedge d^c(\phi - \psi) \wedge T$$

where

$$T = \sum_{j=0}^{n-1} (\omega + dd^c\phi)^j \wedge (\omega + dd^c\psi)^{n-1-j}$$

is a closed positive $(n-1, n-1)$-form.

We will discuss the existence of ϕ under the constraint $\int_M \phi\omega^n = 0$. The idea of the proof is to show that the set S of $t \in [0,1]$ such that there exists $\phi_t \in C^{k+2,\alpha}(M)$ with $\int_M \phi_t\omega^n = 0$ that satisfies

$$(\omega + dd^c\phi_t)^n = (tf + 1 - t)\omega^n \tag{4}$$

is both open and close in $[0,1]$, therefore is the entire interval as $0 \in S$.

To see that S is open, one only has to prove that the function $\mathcal{N}$ defined by

$$\phi \mapsto \mathcal{N}(\phi) = \frac{\det(h_{i\bar{j}} + \phi_{i\bar{j}})}{\det(h_{i\bar{j}})}$$

or in other words $(\omega + dd^c\phi)^n = \mathcal{N}(\phi)\omega^n$, is a local diffeomorphism. The differential of $\mathcal{N}$ is given by

$$D\mathcal{N}(\phi).\eta = \mathcal{N}\Delta\eta$$

with η varies in $\{\eta \in C^{k,\alpha}(M) : \int_M \eta\omega^n = 0\}$, and Δ is the Laplace-Beltrami operator which is known (see [War83, Chapter 6] or [GT83, Chapter 8] for example) to be bijective between

$$\left\{\eta \in C^{k+2,\alpha}(M) : \int_M \eta = 0\right\} \longrightarrow \left\{f \in C^{k,\alpha}(M) : \int_M f = 0\right\}$$

Therefore $\mathcal{N}$ is a local diffeomorphism and S is open.

The proof that S is closed is more technical and is accomplished in 3 steps:

1. Using Arzela-Ascoli theorem, it suffices to show that $\{\phi_t : t \in S\}$ is bounded in $C^{k+2,\alpha}$. In fact if ϕ_{t_n} solves (4) for $t = t_n$ converging to $\tau \in [0,1]$ then up to a subsequence, one can suppose that $\{\phi_{t_n}\}$ converges in $C^{k+1,\alpha}$ to $\phi_\tau \in C^{k+1,\alpha}$ that solves (4) for $t = \tau$. The fact that $\phi_\tau \in C^{k+2,\alpha}$ is due the following Schauder estimate.

2. Using Schauder theory, one can prove the following estimate for any ϕ satisfying (3):
$$\phi \in C^{2,\alpha}, f \in C^{k,\alpha} \implies \phi \in C^{k+2,\alpha}, \text{ and } \|\phi\|_{k+2,\alpha} \leq C$$
where C depends only on M and the upper bounds for $\|\phi\|_{2,\alpha}, \|f\|_{k,\alpha}$.

3. It remains then to prove the following *a priori estimate*:
   There exists $\alpha \in (0,1)$ and $C(M, \|f\|_{1,1}, 1/\inf_M f) > 0$ such that every $\phi \in C^4(M)$ satisfying $(\omega + dd^c\phi)^n = f\omega^n$, $\omega + dd^c\phi > 0$ and $\int_M \phi\omega^n = 0$ (we will call such ϕ *admissible*) has
$$\|\phi\|_{2,\alpha} \leq C.$$

To achieve the a priori estimate, one firstly bounds ϕ in C^0, then bound $\|\Delta\phi\|$ and finally establishs the $C^{2,\alpha}$ estimate. We will give here some details of the first step. For more detail, see [Bł12].

Proof of the C^0-estimate. Since ϕ is defined up to an additive constant, what we mean by the C^0 -estimate for ϕ is in fact the estimate of

$$\mathrm{osc}_M\phi := \max_M \phi - \min_M \phi$$

by a constant C that depends only on M and f. Without loss of generality, one assumes that $\int_M \omega^n = 1$ and $\max_M \phi = -1$. Therefore $\|\phi\|_{L^p} \leq \|\phi\|_{L^q}$ for $p \leq q < \infty$.

One has for every $p \geq 1$:

$$\int_M (-\phi)^p(f-1)\omega^n = \int_M (-\phi)^p dd^c\phi \wedge \left(\sum_{j=0}^{n-1}(\omega + dd^c\phi)^j \wedge \omega^{n-1-j}\right) \tag{5}$$
$$= p\int_M (-\phi)^{p-1} d\phi \wedge d^c\phi \wedge \left(\omega^{n-1} + \sum_{j=1}^{n-1}(\omega + dd^c\phi)^j \wedge \omega^{n-1-j}\right) \tag{6}$$
$$\geq p\int_M (-\phi)^{p-1} d\phi \wedge d^c\phi \wedge \omega^{n-1} \tag{7}$$
$$= \frac{4p}{(p+1)^2}\int_M d(-\phi)^{(p+1)/2} \wedge d^c(-\phi)^{(p+1)/2} \wedge \omega^{n-1} \tag{8}$$
$$= \frac{c_n p}{(p+1)^2}\|D(-\phi)^{(p+1)/2}\|_{L^2}^2 \tag{9}$$

where we used the fact that $\omega + dd^c\phi > 0$ in the inequality, and c_n is a constant depending only on n.

Now we use the following Sobolev inequality on M (i.e. use Sobolev inequality in each chart as a domain of $\mathbb{R}^m$ then add up the results):

$$\|v\|_{L^{mq/(m-q)}} \leq C(M,q)(\|v\|_{L^q} + \|Dv\|_{L^q}), \quad \forall v \in W^{1,q}(M), q < m$$

with $v = \phi$, $m = 2n$ the real dimension of M and $q = 2$, then use (9) to bound the $D\phi$ term:

$$\|(-\phi)^{(p+1)/2}\|_{L^{2n/(n-1)}} \leq C_M\left[\|(-\phi)^{(p+1)/2}\|_{L^2} + \frac{p+1}{\sqrt{p}}\left(\int_M (-\phi)^p(f-1)\omega^n\right)^{1/2}\right]$$

Replace $p+1$ by p and use the fact that $|\phi| \geq 1$, one has

$$\|\phi\|_{L^{np/(n-1)}} \leq (Cp)^{1/p}\|\phi\|_{L^p}, \quad \forall p \geq 2$$

where C depends only on M and $\|f\|_{L^\infty}$.

Repeatedly apply this inequality (this technique is called *Moser's iteration*) one has $\|\phi\|_{L^{p_{k+1}}} \leq (Cp_k)^{1/p_k}\|\phi\|_{L^{p_k}}$ where the sequence p_k is defined by $p_0 = 2$ and $p_{k+1} = \frac{n}{n-1}p_{k-1} = 2(\frac{n}{n-1})^k$ and

$$\|\phi\|_{L^\infty} = \lim_{k\to\infty} \|\phi\|_{L^{p_k}} \leq \|\phi\|_{L^2} \prod_{j=0}^{\infty}(Cp_j)^{1/p_j}$$

with $\prod_{j=0}^{\infty}(Cp_j)^{1/p_j} = (n/(n-1))^{n(n-1)/2}(2C)^{n/2}$

Moreover, one can bound $\|\phi\|_{L^2}$ with $\|\phi\|_{L^1}$ as follow. Using (9) for $p = 1$, one has

$$(\|f\|_{L^\infty} + 1)\|\phi\|_{L^1} \geq \int_M |\phi|(f-1)\omega^n \geq \frac{c_n}{4}\|D\phi\|^2_{L^2}$$

Using Poincaré inequality, one has $\|D\phi\|^2_{L^2} \geq C'(M)\left(\|\phi\|^2_{L^2} - (\int_M \phi\omega^n)^2\right)$. Therefore

$$\|\phi\|_{L^2} \leq C(M, \|f\|_{L^\infty})(\|\phi\|_{L^1} + 1)$$

It remains to prove that $\|\phi\|_{L^2}$ is bounded, which is the following lemma. □

Lemma 15 (L^1-boundedness)**.** *For any admissible ϕ with $\max_M \phi = -1$ one has $\|\phi\|_{L^1} \leq C(M)$*

Proof. Let ψ be the local potential of the Kähler form ω, i.e. a function defined on each chart (not necessarily agrees on zones where charts are glued together) such that $\omega = dd^c\psi = \frac{\sqrt{-1}}{2}h_{i\bar{j}}dz_i \wedge d\bar{z}_j$ where $h_{i\bar{j}}$ can be intepreted as $\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\psi$. We also suppose that the function ψ is negative on every chart. The fact that $\omega + dd^c\phi > 0$ is rewritten as $(h_{i\bar{j}} + \phi_{i\bar{j}}) > 0$ in local coordinates.

Note $u = \psi + \phi$ the potential of $\omega + dd^c\phi$ locally defined on each chart, then u is negative and plurisubharmonic (psh). For every $x \in B(y, R)$ one has

$$u(x) \leq \frac{1}{\mathrm{vol}(B(x, 2R)} \int_{B(x,2R)} u \leq \frac{1}{\mathrm{vol}(B(y, 2R))} \int_{B(y,R)} u$$

where the first inequality is due to plurisubharmonicity and the second is due to $u \leq 0$. Therefore

$$\|u\|_{L^1(B(y,R))} \leq \mathrm{vol}(B(y, 2R)) \inf_{B(y,R)} |u|,$$

hence

$$\|\phi\|_{L^1(B(y,R))} \leq \|u\|_{L^1(B(y,R))} \leq \mathrm{vol}(B(y, 2R))(\inf_{B(y,R)} |\phi| + \max_M |g|).$$

To see that $\|\phi\|_{L^1}$ is bounded, we apply the following Lemma 16 to the covering of M by finitely many balls $B(y_i, R_i)$, $c_i = \mathrm{vol}(B(y_i, 2R_i))$, $d_i = c_i \max_M |g|$ and $r = 1$. □

Lemma 16 (Combinatoric)**.** *Let M be a connected compact manifold covered by finitely many local charts $\{V_i\}_{i=1}^l$ and $r, c_i, d_i > 0$. Then for any continuous function ϕ globally defined on M such that*

$$\|\phi\|_{L^1(V_i)} \leq c_i \inf_{V_i} |\phi| + d_i, \quad \min_M |\phi| \leq r,$$

one has $\|\phi\|_{L^1} := \sum_i \|\phi\|_{L^1(V_i)} \leq C(\{V_i\}, \{c_i\}, \{d_i\}, r)$

Proof. Let p be a point in M where $|\phi|$ attains its minimum. Since M is connected, for every V_i, there exists a sequence $V_{i_k}, 0 \leq k \leq l$ such that

$$i_0 = i, \quad V_{i_k} \cap V_{i_{k+1}} \neq \emptyset, \quad p \in V_{i_l}.$$

One has

$$\begin{aligned}\|\phi\|_{L^1(V_{i_k})} &\leq c_{i_k} \inf_{V_{i_k}} |\phi| + d_{i_k} \leq c_{i_k} \inf_{V_{i_k} \cap V_{i_{k+1}}} |\phi| + d_{i_k} \\ &\leq c_{i_k} \frac{1}{\operatorname{vol}(V_{i_k} \cap V_{i_{k+1}})} \|\phi\|_{L^1(V_{i_{k+1}})} + d_{i_k}.\end{aligned}$$

Repeatedly apply this inequality for $k = 0, \ldots, l-1$, one has

$$\begin{aligned}\|\phi\|_{L^1(V_i)} &\leq A(i, \{V_j\}, \{c_j\}, \{d_j\}) \|\phi\|_{L^1(V_{i_l})} + B(i, \{V_j\}, \{c_j\}, \{d_j\}) \\ &\leq A(i, \{V_j\}, \{c_j\}, \{d_j\})(c_{i_l} r + d_{i_l}) + B(i, \{V_j\}, \{c_j\}, \{d_j\}).\end{aligned}$$

Take the sum for all $i = 0, \ldots, l$ and the result follows. $\square$

4.4 Calabi-Yau manifold

Recall that we defined a Calabi-Yau manifold to be a compact Riemannian manifold of dimension $2n$ with holonomy contained in $SU(n)$. We also remark, using parallel transport, the existence of a compatible complex structure ($U(n)$ suffices) and a holomorphic form non-vanishing at every point. We present here some equivalent definitions of compact Calabi-Yau manifolds.

Theorem 17. *Let X be a compact manifold of Kähler type and complex dimension n then:*

1. *The followings are equivalent*
   - (a) *There exists a Kähler metric such that the global holonomy is in $SU(n)$.*
   - (b) *There exists a holomorphic $(n, 0)$ form that vanishes nowhere.*
   - (c) *The canonical bundle K_X is trivial.*
   - (d) *The structure group of TX can be reduced to $SU(n)$.*
2. *The following are equivalent. If X is simply-connected, they are equivalent with the 4 statements above.*
   - (a) *There exists a Kähler metric such that the local holonomy is in $SU(n)$.*
   - (b) *The canonical bundle K_X is flat.*
   - (c) *There exists a Kähler metric such that the Ricci curvature vanishes.*
   - (d) *The first Chern class vanishes.*

The proof is straightforward with the only non-trivial part is when one needs Calabi-Yau theorem to construct Ricci-flat metric.

5
CHEEGER-GROMOLL SPLITTING

We will prove the following result by Cheeger and Gromoll by a slightly modified approach of Besse in [Bes07, Theorem 6.79, page 172].

Theorem 18 (Cheeger-Gromoll)**.** *Let M be a complete, connected Riemannian manifold with non-negative Ricci curvature. Suppose that M contains a line then M is isometric to $M' \times \mathbb{R}$ with M' a complete, connected Riemannian manifold with non-negative Ricci curvature.*

Note that the notion of geodesic ray or geodesic line used here is rather strict: A geodesic line γ is a geodesic parameterized by $\mathbb{R}$ such that the distance between two point is exactly the distance *on the geodesic*, for example, geodesic line, if it passes by $p \in M$ with velocity v of norm 1, satisfies

$$d(exp_p(tv), exp_p(-sv)) = s + t, \quad \forall s, t > 0$$

5.1 Busemann function

Let γ be a geodesic ray. We construct the Busemann function b associated to the ray as

$$b(x) = \lim_{t \to +\infty} (t - d(x, \gamma(t)))$$

where the limit exists because the sequence $f_t : \ x \mapsto t - d(x, \gamma(t))$ is non-decreasing and bounded above by $d(x, \gamma(0))$. The convergence is also uniform on every compact of M.

In Euclidean space for example, the Busemann function is the orthogonal projection on γ. We will see that in a Riemannian manifold with non negative curvature, the Busemann function will serve as a projection.

Now with a fixed $x_0 \in M$, the tangent vectors at x_0 of the geodesics connecting x_0 and $\gamma(t)$ is in the unit sphere of $T_x M$, which is compact. Let X be a limit point of these tangents vectors, we defined

$$b_{X,t}(x) = b(x_0) + t - d(x, exp_{x_0}(tX))$$

where $exp_{x_0}(tX)$ is the geodesic starting at x_0 with velocity X.

Remark 9. 1. *From the construction of X, one has $b(x_0) + t = b(exp_{x_0}(tX))$, therefore $b_{X,t} \leq b$ with equality in x_0. We say that b is supported by $b_{X,t}$ at x_0. In general a function f is supported by g at x_0 if $f(x_0) = g(x_0)$ and $f \geq g$ in a neighborhood of x_0.*

2. *$b_{X,t}$ is smooth and a computation in local coordinate gives $\Delta b_{X,t}(x_0) \leq \frac{\dim M - 1}{t}$, where Δ denotes the Laplacian with repsect to the metric on M.*

3. $\|\nabla b_{X,t}\| = 1$.

The estimation given on the second point of Remark 9 is established using Jacobi fields:

Lemma 19. *Let M be a Riemannian manifold of dimension n with non-negative Ricci curvature. The function $f(x) = d(x, x_0)$ satisfies at any point x out of the cut-locus of x_0:*

$$\Delta f(x) \geq -\frac{n-1}{f(x)}$$

where Δ denotes the Laplacian with repsect to the metric on M.

Proof. Let $N(t), 0 \leq t \leq l$ be the velocity of the geodesic γ from x_0 to x, and $E_1, \dots, E_{n-1}, N$ be a parallel frame along γ. Let J_i be the unique Jacobi fields along γ with $J_i(l) = E_i(l)$ and $J_i(0) = 0$ (existence and uniqueness of J_i is due to the fact that x is not in the cut-locus).

Then basic manipulation of Jacobi fields gives (without the fact that curvature is non-negative):

$$-\Delta f(x) = \int_0^l dt \sum_{i=1}^{n-1} (< \nabla_N J_i, \nabla_N J_i > - < R(N, J_i)J_i, N >) = \sum_{i=1}^{n-1} I_\gamma(J_i, J_i)$$

where I_γ is the index form of γ. Note that the Jacobi fields J_i coincide with the fields $\frac{t}{l}E(t)$ at 0 and l, therefore by the *fundamental inequality* of index form (see [Sak96, Lemma 2.10 page 95] for details about Jacobi fields and Fundamental inequality of index form):

$$I_\gamma(J_i, J_i) \leq I_\gamma(\frac{t}{l}E_i, \frac{t}{l}E_i)$$

hence

$$-\Delta f(x) \leq \int_0^l \sum_{i=1}^{n-1} < \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i > - < R(N, \frac{t}{l}E_i)\frac{t}{l}E_i, N >$$

The curvature term being $\frac{t^2}{l^2}Ric(N, N)$ non-negative, one has

$$-\Delta f(x) \leq \int_0^l dt \sum_{i=1}^{n-1} < \nabla_N \frac{t}{l}E_i, \nabla_N \frac{t}{l}E_i >= \frac{n-1}{l}.$$

□

We also note that for Theorem 18 it suffices to show that the Busemann function b is harmonic. Let us give some details. In fact, from the smoothness one has $\nabla b(x_0) = \nabla b_{X,t}(x_0)$, which means $\|\nabla b\| = 1$ at every point in M. For each point $y \in M$, there exists a unique x with $b(x) = 0$ and time t for which the flow of ∇b starting at x arrives at y at time t. M is therefore homeomorphic to $\bar{M} \times \mathbb{R}$ by the map $F : y \mapsto (x, t)$. In order that F is an isometry, it suffices to prove that the gradient field ∇b is parallel.

In fact, $\bar{M}$ being equiped with the restriction of the metric on M, the fact that F is isometric is equivalent to the the fact that the flow Φ^t of ∇b is isometric for every time t, which means $\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u >$ vanishes at $t = 0$. But

$$\frac{d}{dt} < \Phi^t_* u, \Phi^t_* u > |_{t=0} = 2 < \nabla_{\partial t} \Phi^t_* u, u > |_{t=0} = 2 < \nabla_u \nabla b, u >$$

where in the second equality we used Schwarz lemma for commuting derivatives of $\Phi(t, x) = \Phi^t(x)$. The vanishing of $< \nabla_u \nabla b(x), u >$ for every vector u is, by bilinearity, equivalent to that of $\nabla_u \nabla b$ for every u, meaning that ∇b is parallel.

The fact that ∇b is parallel is due to a simple computation:

$$Ric(N, N) = N(\Delta b) - \|\nabla N\|^2$$

where $\|\nabla N\|^2 = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} < \nabla_{E_i} N, E_j >^2$. We see that $N = \nabla b$ is parallel if $\Delta b = 0$, as the Ricci curvature is non-negative.

Remark 10. *1. One can show (see [Bes07, Lemma 6.86, page 176]) that every gradient field ∇b of norm 1 at every point is actually harmonic.*

2. Using de Rham decomposition, one has directly the splitting of M if it is simply-connected since N is parallel and M is complete.

5.2 Harmonicity

The Busemann function associated to a geodesic ray is subharmonic, it is a consequence of the following lemma.

Lemma 20. *On a connected Riemannian manifold, if a continuous function f is supported at any point x by a family of C^∞ functions $f_{x,\epsilon}$ depending on x and $\epsilon \to 0$ with $\Delta f_{x,\epsilon}(x) \leq \epsilon$, then f can not attains local strict maximum.*

Proof. Suppose that f attains its strict maximum at a point x_0 and let B be a small geodesic ball containing x_0. Suppose that we have a function h on B with $\Delta h < 0$ on B and $|h|$ small enough such that $f + h$ attains maximum at x in the interior of B. Then $f_{x,\epsilon} + h$ also attains maximum at x, which means $\Delta f_{x,\epsilon} + \Delta h \geq 0$ at x. Let $\epsilon \to 0$, we have a contradiction.

For the construction of the function h, one suppose that B is small enough such that $f|_{\partial B} \leq \max_B f =: f(x_0)$ and equality is not attained at all points of ∂B. Then choose

$$h = \eta(e^{\alpha\phi} - 1)$$

with $\phi(x) = -1$ if $x \in \partial B$ and $f(x) = f(x_0)$, $\phi(x_0) = 0$, $\nabla \phi \neq 0$ and a constant α. We can find α large enough such that

$$\Delta h = \eta(-\alpha^2 \|\nabla \phi\| + \alpha \Delta \phi) e^{\alpha\phi}$$

is negative. □

Now Let us prove the subharmonicity of the Busemann function b. Given a harmonic function h that coincides with b in the boundary ∂B of a geodesic ball B, then $b-h$ is supported by $b_{X(x_0),t}-h$ at any point $x_0 \in B$, where $X(x_0)$ is a unit tangent vector at x_0. By Lemma 19, one has $\Delta(b_{X(x_0),t}-h)(x_0) = \frac{n-1}{t} \to 0$ as t tends to $+\infty$. By Lemma 20 $b-h$ can not attain strict maximum in the interior of B, hence $b-h \leq (b-h)|_{\partial B} = 0$ in B. We have just proved the following lemma:

Lemma 21. *The Busemann function of a geodesic ray in a Riemannian manifold M with non-negative Ricci curvature is subharmonic.*

Now let b_+ be the function previously constructed for the ray $\gamma|_{[0,+\infty[}$ and b_- the Busemann function for the ray $\tilde{\gamma}|_{[0,+\infty[}$ where $\tilde{\gamma}(t) = \gamma(-t)$. Note that $b_+ + b_- \leq 0$ with equality on the line γ, but the sum is subharmonic therefore by maximum principle $b_+ + -b_- = 0$ and b is harmonic therefore smooth. The splitting theorem of Cheeger-Gromoll follows.

5.3 Application

A consequence of Theorem 18 is the following result from [CG71] (Theorem 2)

Theorem 22. *Let M be a compact Riemannian manifold with non-negative Ricci curvature, then the universal covering space of M is of form $\mathcal{M} = \mathbb{R}^n \times \bar{M}$ where $\bar{M}$ does not contain any lines. Then $\bar{M}$ is compact.*

Proof. It suffices to prove that if $\bar{M}$ is not compact, then it contains a line. In fact, it is easy to see that such $\bar{M}$ must contains a (strict) geodesic ray. In fact it is obvious that for a fixed $p \in M$ the function

$$F : v \mapsto \inf\{t > 0 : d(p, exp_p(tv)) < t\}$$

defined on the unit ball U_p of $T_p\bar{M}$ is upper semi-continuous. Therefore if $F(v) < \infty$ for all unit tangent vector v at p then F is bounded above in U_p by a constant c. Therefore $\bar{M} \subset exp_p(cU_p)$ which is compact (contradiction). Therefore there exists a minimal ray at every point $p \in \bar{M}$.

The existence of a line in general might not be true, the only extra property of $\bar{M}$ that we will need is that it has a (fundamental) domain K compact such that $\bar{M} = \bigcup_{\sigma \in Isom(\bar{M})} \sigma K$.

Let us first prove that such domain K exists. Remark that every isometry of $\mathcal{M}$ acts separately on $\mathcal{M}$, i.e. of form $\sigma(u) = (\sigma^1(x), \sigma^2(y))$ for $u = (x, y) \in \mathcal{M}$ with σ^1, σ^2 isometries of $\mathbb{R}^n$ and $\bar{M}$. This can be seen by the uniqueness part of de Rham decomposition or simply by noticing that a tangent vector in the $T_x\mathbb{R}^n$ component is characterized by the fact that its geodesics is a line. As the action of $\pi_1(M)$ on $\mathcal{M}$ is free and proper, it has a fundamental domain H. We then can choose K to be the projection of H on $\bar{M}$ as $\bar{M} = \bigcup_{\sigma=(\sigma^1,\sigma^2)\in\pi_1(M)} \sigma^2(H)$.

Now let γ be a minimal ray starting from $p \in M$, for each $x \in \gamma$ there exists an isometry σ of $\bar{M}$ such that $\sigma(x) \in K$. By compactness of K, there exists a sequence

$t_n \to +\infty$ with $x_n = \gamma(t_n)$, $v_n = \dot{\gamma}(t_n)$ that satisfies $y_n = \sigma_n(x_n) \to y \in K$ and even more, $(\sigma_n)_* v_n \to v \in T_y\bar{M}$ in the tangent bundle $T\bar{M}$. Then the geodesic of $\bar{M}$ starting at y with vector v is a line. In fact it suffices to prove that $d(exp_y(tv), exp_y(-sv)) = s+t$ for $s, t > 0$, but for n large enough that $t_n > s$ one has

$$d(exp_{y_n}(tv_n), exp_{y_n}(-sv_n)) = s + t$$

then let $n \to +\infty$, one sees that $\bar{M}$ contains a geodesic line, which is contradictory. $\square$

6
BOGOMOLOV-BEAUVILLE DECOMPOSITION

6.1 From the Riemannian results of de Rham and Berger

We will first prove a (conceptually) straightforward result of de Rham decomposition and Berger classification. The following theorem is taken from Beauville's article.

Theorem 23 (Beauville). *Let X be a compact Kähler manifold with flat Ricci curvature, then*

1. *The universal covering space $\tilde{X}$ of X decomposes isometrically and holomorphically as*
$$\tilde{X} = E \times \prod_i V_i \times \prod_j X_j$$
where $E = \mathbb{C}^k$, V_i and X_j are simply-connected compact manifolds of real dimension $2m_i$ and $4r_j$ with irreducible homonomy $SU(m_i)$ for V_i and $Sp(r_j)$ for X_j. One also has uniqueness in the strong sense as in de Rham decomposition.

2. *There exists a finite covering space X' of X such that*
$$X' = T \times \prod_i V_i \times \prod_j X_j$$
where T is a complex torus.

Proof. Note that the first point is obtained directly from Cheeger-Gromoll splitting and de Rham decomposition: The one-dimensional parallel subspaces (of trivial holonomy) are regrouped to E. By Cheeger-Gromoll splitting, $\tilde{X} = E \times M$ where M contains no line and is compact (note that we use compactness of X here). The irreducible factors in M are not symmetric spaces as Ricci curvature of symmetric spaces is non-degenerate. Holonomy of these factors are $SU(m_i)$ and $Sp(r_j)$ according to Berger list since they are Kähler manifolds and Ricci-flat. It remains to prove the second point.

We will regard each element of $\pi_1(X)$ by its isometric, free, proper action on $\tilde{X}$. As pointed out the arguments in our discussion of uniqueness of de Rham decomposition, every isometry of $\tilde{X}$ to itself preserves the components $T_{x_0}E$, $T_{x_i}V_i$ and $T_{x_j}X_j$ of $T_x\tilde{X}$, each isometry ϕ of $\tilde{X}$ is of form (ϕ_1, ϕ_2) where $\phi_1 \in Isom(E)$ and $\phi_2 \in Isom(M)$.

We will use here the fact that if M is a Kähler manifold, compact and Ricci-flat then $Isom(M)$ equipped with compact-open topology is discrete, therefore finite, which will be proved later (Lemma 25). We note $\Gamma := \{\phi = (\phi_1, \phi_2) \in \pi_1(X),\ \phi_2 = Id_M\}$ and sometime abusively regard Γ as a subgroup of $Isom(E)$. Note that Γ is a normal subgroup of $\pi_1(X)$ with finite index since the quotient is isomorphic to a subgroup of $Isom(M)$. Therefore $\tilde{X}/\Gamma = E/\Gamma \times M$ is compact as a finite cover of X.

We apply the following theorem of Bieberbach.

Theorem 24 (Bieberbach)**.** *Let $E = \mathbb{R}^n$ be an Euclidean space and Γ be a subgroup of $Isom(E)$ that satisfies*

1. *Γ is discrete under compact-open topology.*
2. *E/Γ is compact.*

Then the subgroup Γ' of translations in Γ is of finite index.

Suppose that the two conditions are satisfied then the theorem gives: $\tilde{X}/\Gamma' = E/\Gamma' \times M = T \times \prod_i V_i \times \prod_j X_j$ is a finite cover of $\tilde{X}/\Gamma$ as Γ' is a normal subgroup of Γ:

Fact. The subgroup of translations in $Isom(E)$, where $E = \mathbb{R}^n$ is an Euclidean space, is normal.

Therefore $X' = \tilde{X}/\Gamma'$ is a finite cover of X that we want to find.

It remains to prove that Γ is discrete, which is a consequence of

1. $\pi_1(X)$ is discrete, without limit point in $Isom(E) \times Isom(M)$ (obvious).
2. $Isom(M)$ is compact.

In fact given any $\phi = (\phi_1, \phi_2) \in Isom(E) \times Isom(M)$, there exists by (1.) a neighborhood $\mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$ of ϕ in $Isom(E) \times Isom(M)$ such that all points of $\pi_1(X)$ lying in this region project to ϕ_1. By (2.) we can find a neighborhood $\mathcal{U}_1$ of ϕ_1 in $Isom(E)$ small enough such that $\mathcal{U}_1(\phi_1) \times Isom(M) \subset \cup_{\phi_2 \in Isom(M)} \mathcal{U}_1(\phi_1, \phi_2) \times \mathcal{U}_2(\phi_1, \phi_2)$. Therefore the projection of $\pi_1(X)$ to $Isom(E)$ is discrete, by consequence Γ is discrete. □

Lemma 25. *Let M be is a compact, simply-connected, Ricci-flat, Kähler manifold, then the group $Aut(M)$ of automorphism of M equipped with compact-open topology is discrete, therefore $Isom(M)$ is discrete, hence finite.*

Proof. The idea is that since $Aut(M)$ is a Lie group, it suffices to prove that its Lie algebra is of dimension 0. This is done using these facts.

1. The Lie algebra of $Aut(M)$ can be identified with the vector space of holomorphic vector fields on M.
2. *Bochner's principle*: All holomorphic tensor fields on a compact, Ricci-flat Kähler manifold are parallel.
3. The only invariant vector of the holonomy representation of M is 0 (obvious).

□

Bochner principle for holomorphic vector fields comes from the following identity (called *Weitzenbock formula*):

$$\Delta(\frac{1}{2}\|X\|^2) = \|\Delta X\|^2 + g(X, \nabla \mathrm{div} X) + Ric(X, X)$$

for every vector field X. If X is holomorphic then it is harmonic and has $\mathrm{div}X = 0$. The fact that M is Ricci-flat gives $\Delta(\frac{1}{2}\|X\|^2) = \|\nabla X\|^2$ and the function $\|X\|^2$ is subharmonic, therefore constant since M is compact. We then have $\nabla X = 0$,i.e. X is parallel. The method of Bochner also works for tensor fields of any type in a Ricci-flat Kähler manifold and one also obtains $\Delta(\|\tau\|^2) = \|\nabla\tau\|^2$ and that every holomorphic tensor field is parallel. See [Pet06, Chapter 7] and [Bes07, Paragraph 1.156 page 57, Lemma 14.17 page 399] for more details.

6.2 Towards a classification for complex manifold

To obtain a translation of Theorem 23 in a context of complex manifolds (without any preferred metric a priori), we study the 2 building blocks: manifolds with holonomy $SU(m)$ and $Sp(r)$. To be clear, recall that a complex manifold X is called of Kähler type if one can equip X with an Hermitian structure whose fundamental form ω satisfies $d\omega = 0$. When we say X is of Kähler type, we refer to X as a complex manifold without fixing a metric on X.

6.2.1 • Special unitary manifolds (proper Calabi-Yau manifolds)

Remark 11. *Let X be a compact Kähler manifold with holonomy $SU(m)$ and complex dimension $m \geq 3$ then:*

1. $H^0(X, \Omega^p_X) = 0$ *for all* $0 < p < m$ *and consequently* $\chi(\mathcal{O}_X) = 1 + (-1)^m$.
2. *X is* projective*, i.e. X can be embedded into $\mathbb{CP}^N$ as zero-locus of some (finitely) homogeneous polynomials.*
3. $\pi_1(X)$ *is finite and if m is even, X is simply connected.*

The first point is in fact algebraic in nature: it comes from the fact that the representation of $SU(m)$ over $\bigwedge^p T^*_x M$ is irreducible for all p et non-trivial for $0 < p < m$, therefore the action of $SU(m)$ on $\bigwedge^p T^*_x M$ for $0 < p < m$ has no invariant element, hence $H^0(X, \Omega^p_X) = 0$.

The second point follows from the following non trivial facts that we shall admit:

1. (Kodaira embedding theorem) A compact Kähler manifold with $H^{2,0} = 0$ can be embedded in $\mathbb{P}^N$.

2. (Chow's theorem) A compact complex manifold embedded in $\mathbb{P}^N$ is algebraic, i.e. defined by a finite number of homogeneous polynomials.

The third point is a consequence of Riemann-Hurwitz formula. In fact, the universal cover $\tilde{X}$ of X is of holonomy $SU(m)$. This is due to the following remarks: $Hol(X) \supset Hol(\tilde{X}) \supset Hol_0(\tilde{X}) = Hol_0(X)$ and $Hol_0(X)$ is the identity component of $Hol(X)$, which is also $Hol(X)$ as $SU(m)$ is connected.

By Theorem 23, $\tilde{X}$ is compact as its holonomy has no trivial component. By Lemma 25, $\pi_1(X)$ is finite therefore $\tilde{X}$ is a finite cover of X. As $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}}) = 2$, one has $X = \tilde{X}$, hence X is simply-connected.

Theorem 26. *Given a compact manifold X of Kähler type and complex dimension m, the following properties are equivalent*

1. *There exists a compatible metric g over X such that $Hol(X, g) = SU(m)$.*

2. *K_X is trivial and $H^0(X', \Omega^p_{X'}) = 0$ for every $0 < p < m$ and any X' finite cover of X.*

Proof. (1) implies (2) as a finite covering space X' of a special unitary manifold X is still a special unitary.

For the implication (2) $\implies$ (1): by Yau's theorem we equip X with a Ricci-flat metric, by Theorem 23, there exists a finite cover $X' = T \times \prod_i V_i \times \prod_j X_j$ where T is a complex torus, $Hol(V_i) = SU(m_i), Hol(X_j) = Sp(r_j)$. But $H^0(X', \Omega^p_{X'}) = 0$ for $0 < p < m$, X' has to be one of the V_i as $H^0(X_j, \Omega^2_{X_j})$ and $H^0(V_i, \Omega^{m_i}_{V_i})$ do not vanish. Therefore $Hol(X') = SU(m)$, hence $Hol(X) = SU(m)$. □

Theorem 26 allows us to check if a manifold X is special unitary by looking at the $h^{0,p}(0 < p < m)$ coefficients of the Hodge diamond of X and its finite covers. We can see, by this criteria that the following examples are special unitary manifolds.

Example 1 (Special unitary manifolds)**.**

1. *Elliptic curves over $\mathbb{C}$ are special unitary, as any statement starting with "for every $0 < p < 1$" is formally true.*

2. *A K3 surface (simply-connected surface with trivial canonical bundle) is special unitary, its Hodge diamond is given below. K3 surface may not be projective but are Kähler.*

3. *A quintic threefold (hypersurface of degree 5 in 4-dimensional projective space) is a special unitary manifold, the Hodge diamond of which is given is given below. In particular, the Fermat quintic defined by*

$$\{(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^4 : \sum z_i^5 = 0\}$$

4. *In general, any smooth hypersurface X of $\mathbb{CP}^{m+1}$ of degree $m+2$ satisfies $h^{0,p} = 0$ for all $0 < p < m$. If X is simply-connected then it is a special unitary manifold.*

Table 1: Hodge diamond of a K3 surface.

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Table 2: Hodge diamond of a quintic threefold.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

6.2.2 • Irreducible symplectic and hyperkähler manifolds

Remark 12. *Let X be a compact Kähler manifold with holonomy $Sp(r)$ and complex dimension $2r$ then:*

1. *There exists a holomorphic 2-form φ non-degenerate at every points.*
2. *$H^0(X, \Omega_X^{2l+1}) = 0, H^0(X, \Omega_X^{2l}) = \mathbb{C}\varphi^l$ for all $0 \le l \le r$. By consequence $\chi(\mathcal{O}_X) = r + 1$.*
3. *X is simply-connected.*

The first point of the remark follows directly from our discussion of Berger classification.

The second point is algebraic in nature: The representation of $Sp(r)$ on $\bigwedge^p T_x^*M$ splits into

$$\bigwedge^p T_x^*M = P_p \oplus P_{p-2}\varphi(x) \oplus P_{p-4}\varphi^2(x) \oplus \dots \tag{10}$$

where $P_k, 0 \le k \le r$ are irreducible, non-trivial representations for $k > 0$ and $\varphi(x) \in \bigwedge^2 T_x^*M$ uniquely defined up to a constant (see [Bou06, § 13, n° 3]). Therefore the only invariant elements are $c\varphi^{p/2}$ where c is a scalar.

For the last point, one uses the same arguments as Remark 11.

Theorem 27. *Given a compact manifold X of Kähler type and complex dimension $2r$, then:*

1. *The following properties are equivalent. X is called hyperkähler if it satisfies one of them.*
   (a) *There exists a compatible metric g such that $Hol(X, g) \subset Sp(r)$.*

(b) *There exists a compatible symplectic structure: a 2-form that is closed, holomorphic and non-degenerate at every point.*

2. *The following properties are equivalent. X is called irreducible symplectic if it satisfies one of them.*

   (a) *There exists a compatible metric g such that $Hol(X, g) = Sp(r)$*

   (b) *X is simply-connected and there exists (uniquely up to a constant) a compatible symplectic structure on X.*

By "compatible", we mean "compatible with the complex structure".

Proof. 1. The fact that (a) implies (b) is obvious. For the other way: since K_X is trivial (existence of global non-null section) by Yau's theorem we equip X with a Ricci-flat metric, then the symplectic structure φ of X is parallel by Bochner's principle. Hence the holonomy is in $Sp(r)$.

2. For the implication (a) $\implies$ (b), it suffices to notice that the invariant elements φ in the decomposition (10) is unique. For the direction (b) $\implies$ (a), note that X can be equipped with a Calabi-Yau metric by the (b) $\implies$ (a) part of (1.). By Theorem 23, $X = \prod_{j=1}^{m} X_j$ where X_j are irreducible compact Kähler manifolds. The symplectic structure φ on X, restricted on each X_j, gives a symplectic structure φ_j of X_j. But any form $\sum_j \lambda_j pr_j^* \varphi_j$ is another symplectic structure of X, one must have $m = 1$ by uniqueness of φ.

□

Example 2. *Only very few examples are known.*

1. *One can notice a trivial example: Every special unitary manifold of 2 complex dimensions is irreducible symplectic because $SU(2)$ is isomorphic to $Sp(1)$.*

2. *Let X be a smooth cubic hypersurface in $\mathbb{CP}^{n+1}$ and*

$$F(X) = \{L \in Gr(1, \mathbb{CP}^{n+1}), L \subset X\} \subset Gr(1, \mathbb{CP}^{n+1})$$

   the manifold formed by lines in X. $F(X)$ is non-empty when $n > 1$, smooth if X is smooth and of dimension $2n - 4$. In [BD85], Beauville and Donagi proved (we admit the proof) that for $n = 4$, $F(X)$ is irreducible symplectic and therefore hyperkähler.

6.2.3 • Decomposition for complex manifold with vanishing Chern class

Theorem 23 can be translated to a decomposition for complex manifold in the following way:

Theorem 28 (Bogomolov-Beauville classification)**.** *Let X be a compact manifold of Kähler type of vanishing first Chern class.*

1. *The universal covering space $\tilde{X}$ of X is isomorphic to a product $E \times \prod_i V_i \times \prod_j X_j$ where $E = \mathbb{C}^k$ and*

   (a) *Each V_i is a projective simply-connected manifold of complex dimension $m_i \geq 3$, with trivial K_{V_i} and $H^0(V_i, \Omega^p_{V_i}) = 0$ for $0 < p < m_i$*

   (b) *Each X_j is an hyperkähler manifold.*

   This decomposition is unique up to an order of i and j.

2. *There exists a finite cover X' of X isomorphic to the product $T \times \prod_i V_i \times \prod_j X_j$.*

The theorem follows directly from Theorem 23, the only point that needs proof is the uniqueness, which will be achieved in two steps:

1. Prove the uniqueness in the case that X is simply-connected.

2. Prove that every isomorphism $\phi : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$ is splitted as $\phi = (\phi_1, \phi_2)$ where $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$ and $\phi_2 : Y \longrightarrow Z$ are isomorphisms (by consequence $h = k$).

These two steps will be accomplished in the following two lemmas

Lemma 29. *Let $Y = \prod_j Y_j$ be a finite product of compact, simply-connected manifold of Kähler type with vanishing Chern class. The Calabi-Yau metrics of Y are then $g = \sum_l pr_j^* g_j$ where g_j are Calabi-Yau metrics of Y_j.*

Proof. Let g be a Calabi-Yau metric of Y and $[\omega]$ its class in $H^{1,1}(Y)$. Since Y_j are simply-connected, $[\omega] = \sum_j pr_j^*[\omega_j]$. By Yau's theorem, there exist unique Calabi-Yau metrics g_j of Y_j in each class $[\omega_j]$. The metric $g' = \sum_j pr_j^* g_j$ is in the same class ω of g and is also a Calabi-Yau metric, hence $g = g' = \sum_j pr_j^* g_j$. □

This lemma asserts that when our manifolds Y, Y_j are equipped with appropriate Calabi-Yau metrics, the decomposition map is also a (Riemannian) isometric, we therefore obtain uniqueness of V_i, X_j from uniqueness of Theorem 23.

Lemma 30. *Let Y, Z be compact, simply-connected manifold of Kähler type, then any isomorphism $u : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z$ is splitted as $\phi = (\phi_1, \phi_2)$ where $\phi_1 : \mathbb{C}^k \longrightarrow \mathbb{C}^h$ and $\phi_2 : Y \longrightarrow Z$ are isomorphisms.*

Proof. It is clear that the composed function $u_1 : \mathbb{C}^k \times Y \longrightarrow \mathbb{C}^h \times Z \longrightarrow \mathbb{C}^h$ is constant in Y, i.e. $u_1(t, y) = u_1(t)$ as holomorphic functions on Y are constant, therefore $u(t, y) = (u_1(t), u_2(t, y))$. As u is isomorphic, one has $h \leq k$ then by the same argument for u^{-1}, one has $h = k$, u_1 is an isomorphism and $u_2(t, \cdot)$ is an isomorphism from Y to Z. $u_2(0, \cdot)^{-1} \circ u_2(t, \cdot)$ is then a curve in $Aut(Y)$, which is discrete by Lemma 25. Therefore $u_2(t, \cdot) = u_2(0, \cdot)$ is independent of t. □

7
FURTHER DEVELOPMENTS

7.1 DECOMPOSITION THEOREM IN CASE OF NON-NEGATIVE RICCI CURVATURE

Theorem 28 is generalized by Campana, Demailly and Peternell in [CDP12] for manifold with *non-negative Ricci curvature* or equivalently, for manifolds with *Hermitian semipositive anticanonical bundle* in the complex point of view.

We noticed that the statement of Theorem 1.4 in [CDP12] was not accurate and we propose a new formulation. Further details will appear somewhere else.

Theorem 31. *Let X be a compact Kähler manifold with K_X^{-1} Hermitian semipositive then the universal cover $\tilde{X}$ decomposes isometrically and holomorphically into*

$$\tilde{X} = \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_l \times \prod W_r$$

where Y_j, S_k, Z_l are compact, simply-connected Kähler manifolds. Y_j are proper Calabi-Yau (holonomy $SU(n_j)$), S_k are hyperkähler (holonomy $Sp(n'_k/2)$), Z_l are rationally connected *with $K_{Z_l}^{-1}$ semipositive (holonomy $U(n''_l)$), and W_r are compact Hermitian symmetric spaces (that are also rationally connected).*

Remark 13. *Each W_r is one of the following Hermitian symmetric spaces*

1. $SU(p+q)/S(U(p) \times U(q))$,
2. $SO(p+2)/SO(p) \times SO(2)$,
3. $SO(2n)/U(n)$,
4. $Sp(n)/U(n)$,
5. $E_6/SO(10) \times SO(2)$,
6. $E_7/E_6 \times SO(2)$.

As symmetrics spaces have positive-definite Ricci curvature ([KN63, Corollary 8.7 page 258, volume 2]), the W_r have positive $K_{W_r}^{-1}$ hence are Fano varieties. By a result of Kollár-Miyaoka-Mori (see [KMM92]), they are rationally-connected.

Sketch of proof. One follows the same strategy by A. Beauville. Compared with Theorem 23,

1. One can not eliminate the irreducible symmetric blocks, since they have positive Ricci curvature.

2. Berger list gives the additional blocks Z_l of holonomy $U(n_l'')$.

One then studies the new blocks Z_l as a complex manifolds, as we did in 6.2.1 and 6.2.2 and proves that a Kähler manifold Z that has a compatible metric under which the holonomy is $U(n)$ is actually rationally connected. $\square$

Remark 14. *We note that unlike Theorem 26 and Theorem 27, the rational connectedness is not a characterization of manifold of holonomy $U(n)$ as a complex manifold. In order words, it is not proved in [CDP12] that a compact, rationally connected, Kähler manifold can be equipped with a compatible metric whose holonomy is $U(n)$.*

It is therefore not a surprise that one can not say anything about uniqueness in Theorem 31.

7.2 Decomposition theorem for singular spaces and klt varieties

The Decomposition Theorem 28 is also generalized for singular spaces in [GKP11] and for klt varieties in [GKP11] by Daniel Greb et al. They are noticeably different with Theorem 28 in the following ways:

1. They are decompositions of tangent sheaf, therefore should be seen as an infinitesimal analogue of Bogomolov-Beauville's results.

2. One does not decompose the tangent sheaf of a finite cover of the manifold X, but rather the tangent sheaf of $\tilde{X}$ such that an existing $A \times \tilde{X}$ is a finite cover of X and A is an Abelian variety.

APPENDICE: PRINCIPAL BUNDLE

Definition and relation with vector bundle

Definition 7. *Given a group G and a manifold M, a* principal bundle $P = (M, G)$ *of fiber G over M is a bundle where*

1. *Each fibre is G (in an atlas).*

2. *Action of G by right-multiplication in local charts is well-defined.*

Here is another way to state the second point: let $\phi_{ij} : G \longrightarrow G$ be the change of fibre from chart j to chart i, the right-multiplication by g is well defined if and only if $\phi_{ij}(h)g = \phi_{ij}(hg)$ which means ϕ_{ij} is a left-multiplication, that we can abusively note by $h \mapsto \phi_{ij}h$ where $\phi_{ij} \in G$ satisfying the *cocycle condition*.

A *homomorphism* $f : P_1 = (M_1, G_1) \longrightarrow P_2 = (M_2, G_2)$ of principal bundles is the data of

1. a group homomorphism $f : G_1 \longrightarrow G_2$.

2. a homomorphism of bundle also noted by $f : P_1 \longrightarrow P_2$ such that $f(ga) = f(g)f(a)$.

In particular, a *principal sub-bundle* of $P = (M, G)$ is a principal bundle $P' = (M, H)$ where H is a subgroup of G such that there exists an injection $f : P' \longrightarrow P$ whose group homomorphism is the injection of H to G. P' can be regarded as a subset of P such that each fiber is H. In this case, one says that P has a H-structure, or equivalently there exists an atlas of P with all $\phi_{ij} \in H$.

We now relate the notion of principal bundle and vector bundle. Given a vector bundle E fibered by a vector space V over a manifold M, we regard E since Riemann as the data of $\phi_{ij} : U_i \cap U_j \times V \longrightarrow U_i \cap U_j \times V$ that satisfy the cocycle condition for a covering of M by open sets U_i. As ϕ_{ij} can also be interpreted as function from $U_i \cap U_j$ to $GL(V)$, a vector bundle is nothing else than a principal bundle of fiber $GL(V)$. The corresponding principal bundle to is called the *frame bundle* of E.

One can also construct a bundle from a principal bundle P of fiber G and a manifold F with action of G on the left (typically a representation of G) by taking

$$P \times_G F = P \times F/(u, v) \sim (ug, g^{-1}v)$$

which is a bundle with fiber F.

Remark 15. 1. *If we take P to be the frame bundle of a vector bundle E of fiber V, and $F = V$ with action of $G = GL(V)$ on the left then*

$$P \times_{GL(V)} V = E$$

2. *We will see later that a connection on P gives a connection on $P \times_G F$, by taking $F = T^r_s(V) = \bigotimes^r V \otimes \bigotimes^s V^*$, we can extend a connection on the tangent fiber to all tensor fiber.*

3. *One has $P \times_G G/H = P/H$ and there is a 1-1 correspondence between the H-structures of P and the global sections of P/H.*

Connection on principal bundle

Definition 8. *A* connection *D on a principal bundle P of fiber G over manifold M is a distribution of subspaces $H_p \subset T_pP$ in every points $p \in TP$ such that*

1. *$T_pP = H_p \oplus V_p$ where V_p is the tangent of the fiber passing by p.*

2. *The plans are stable by G i.e. $H_{gp} = g_* H_p$ for all $p \in P, g \in G$.*

Clearly $V_p = \ker T\pi_{P\to M}$, one has $T\pi_{P\to M} :\ V_p \longrightarrow T_{\pi(p)}M$ is bijective, that means every tangent vector of M at $\pi_{P\to M}(p)$ can be *lifted* to the tangent space at p in a unique way.

Moreover one can also lift a curve in M to P.

Definition 9 (Lemma)**.** *Let $x_0 \in M$ and $p_0 \in P$ such that $\pi_{P\to M} p_0 = x_0$ then for any $\mathcal{C}^1$-curve γ parameterized by $[0, 1]$ in M with $\gamma(0) = x_0$, there exists a unique curve $\tilde{\gamma}$ in P such that $\tilde{\gamma}(0) = p_0$, $\tilde{\gamma}(t)$ is projected to $\gamma(t)$ and $\dot{\gamma}$ is lifted to $\dot{\tilde{\gamma}}$.*

We define $\tilde{\gamma}(1)$ to be the parallel transport *of p_0 by the curve γ and connection D.*

We now relate the previous definitions with what are already seen in the course MAT568 (Relativité générale). It is easy to see that to define a connection as a *covariant derivative* ∇ and parallel transport as the solution of the system $\nabla_{\dot{\gamma}} X = 0$ with $X(0) = p_0$ are the same as to define a connection as the a distribution of plans $H_p \subset T_p(TM)$ of same dimension as M such that $H_{\lambda_1 p_1 + \lambda_2 p_2} = \lambda_1 H_{p_1} + \lambda_2 H_{p_2}$ and to define parallel transport by the *lift* $\bar{\gamma}$ of γ with $\dot{\bar{\gamma}} \in H_p$ (see [Pau14] for example). It remains to see how a distribution of planes in TTM correspond to a distribution of plan in TP when P is the frame bundle of M.

We note $\pi :\ P \times V \longrightarrow P \times_{GL(V)} V = TM$ and remark that and $T_{(p,v)}\pi$ transforms $V_p \oplus T_vV \subset V_p \oplus H_p \oplus T_vV = T_{(p,v)}(P \times V)$ to the vertical plan $V_{\pi(p,v)} \subset T_{\pi(p,v)}TM$ by the following heuristics:

$$\pi_M \pi(p, v) = \pi_M \pi(p(1 + \delta g, v + \delta v) = \pi_M \pi(p, v) + T\pi_M T_{(p,v)}\pi.p\delta g + T\pi_M T_{(p,v)}\pi.\delta v$$

where π_M is the projection from TM to M, $p\delta g \in V_p$ and $\delta v \in T_vV$. By the fact that π is submersion and by dimension, $T_{p,v}\pi$ sends H_p to horizontal plan $H_{\pi(p,v)} \subset T_{\pi(p,v)}TM$.

As an example, we indicate concretely how to transport along a curve γ in M a vector v_0 tangent to M at $x_0 \in \gamma$, knowing the connection on P: We firstly pull back v_0 to a $(p_0, v_0) \in P \times V$ (there are more than one choice, but every choice of (p_0, v_0) gives the same result). We then keep v_0 constant and transport p_0 using the connection of P. We finally project the result to TM using π.

If γ is a closed curve, a point $p_0 \in P$ over $x_0 \in M$ will be transported to $p_0 g$ The elements $g \in G$ obtained by this ways are independent of p_0 and form a group, called the *holonomy group* at x_0 corresponding to connection D. If there is no further indication, we will only consider the holonomy group corresponding to *Levi-Civita connection* on a Riemannian manifold M, which will be denoted by $Hol(M, x)$.

Remark 16. *Note that given a Riemannian manifold M and ∇ its Levi-Civita connection*

1. *We have just defined the holonomy group $Hol(M, x)$ as a subgroup of $GL(T_x M)$, that is by giving a faithful representation of it.*

2. *For any $x, y \in M$ a curve γ that connects x and y gives an isomorphism between $Hol(M, x)$ and $Hol(M, y)$. So the representation/group structure of $Hol(M, x)$ does not depend on x and therefore will be noted by $Hol(M)$.*

3. *Using $\nabla g = 0$, one can see that parallel transport preserves inner product, therefore $Hol(M)$ is contained in $O(n)$ or even $SO(n)$ if M is orientable.*

Here is a concrete interpretation of connection taken from [Ber03] in case of M being a surface in $\mathbb{R}^3$ with the inducted metric. Let γ be a curve on M and X a parallel vector field along γ, if one roll a tangent plan tangent P along the curve and note by γ^* and X^* the images of γ and X traced on P then X^* is a constant field along γ^*.

REFERENCES

[BD85] A. Beauville and R. Donagi. La variété des droites d'une hypersurface cubique de dimension 4. *C.R. Acad. Sc. Paris*, 301:703–706, 1985.

[Bea83] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *Journal of Differential Geometry*, 18(4):755–782, 1983.

[Bea99] A. Beauville. Riemannian Holonomy and Algebraic Geometry. *arXiv:math/9902110*, February 1999.

[Ber03] Marcel Berger. *A Panoramic View of Riemannian Geometry*. Springer Berlin Heidelberg, 2003.

[Bes07] Arthur L. Besse. *Einstein Manifolds*. Springer Berlin Heidelberg, November 2007.

[Bog74] F. A. Bogomolov. Kähler manifold with trivial canonical class. *Mathematics of the USSR-Izvestiya*, 8(1):9, 1974.

[Bou06] Nicolas Bourbaki. *Groupes et algèbres de Lie - Chapitres 7*. Springer-Verlag Berlin Heidelberg, 2006.

[Bus85] Peter Buser. A geometric proof of Bieberbach"s theorems on crystallographic groups. *Enseignement Mathématique (2)*, 31(1-2):137–145, 1985.

[Bł12] Zbigniew Błocki. The Calabi–Yau Theorem. In Vincent Guedj, editor, *Complex Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics*, number 2038 in Lecture Notes in Mathematics, pages 201–227. Springer Berlin Heidelberg, 2012.

[CDP12] Frédéric Campana, Jean-Pierre Demailly, and Thomas Peternell. Rationally connected manifolds and semipositivity of the Ricci curvature. *arXiv:1210.2092 [math]*, October 2012.

[CG71] Jeff Cheeger and Detlef Gromoll. The splitting theorem for manifolds of nonnegative Ricci curvature. *Journal of Differential Geometry*, 6(1):119–128, September 1971.

[CG72] Jeff Cheeger and Detlef Gromoll. On the Structure of Complete Manifolds of Nonnegative Curvature. *Annals of Mathematics*, 96(3):413–443, 1972.

[GGK17] Daniel Greb, Henri Guenancia, and Stefan Kebekus. Klt varieties with trivial canonical class - Holonomy, differential forms, and fundamental groups. *arXiv:1704.01408 [math]*, April 2017.

[GH94] Phillip Griffiths and Joseph Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, Inc., Hoboken, NJ, USA, August 1994.

[GKP11] Daniel Greb, Stefan Kebekus, and Thomas Peternell. Singular spaces with trivial canonical class. *arXiv:1110.5250 [math]*, October 2011.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer Berlin Heidelberg, 1983.

[Huy05] Daniel Huybrechts. *Complex Geometry*. Universitext. Springer-Verlag, Berlin/Heidelberg, 2005. DOI: 10.1007/b137952.

[Joy00] Dominic D. Joyce. *Compact Manifolds with Special Holonomy*. Oxford University Press, 2000.

[KMM92] János Kollár, Yoichi Miyaoka, and Shigefumi Mori. Rational connectedness and boundedness of Fano manifolds. *Journal of Differential Geometry*, 36(3):765–779, 1992.

[KN63] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry*. Interscience Publishers, 1963.

[MZ55] Deane Montgomery and Leo Zippin. *Topological transformation groups*. Interscience Publishers, 1955.

[Pau14] Frédéric Paulin. Groupes et géométries. *Notes de cours*, 2014.

[Pet06] Peter Petersen. *Riemannian Geometry*, volume 171 of *Graduate Texts in Mathematics*. Springer New York, 2006.

[Sak96] Takashi Sakai. *Riemannian Geometry*. American Mathematical Soc., January 1996.

[War83] Frank W. Warner. *Foundations of Differentiable Manifolds and Lie Groups*, volume 94 of *Graduate Texts in Mathematics*. Springer New York, New York, NY, 1983.

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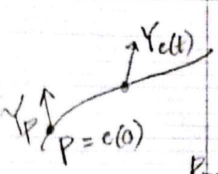
Exo 5 / pg 29



| c : courbe , $c(0) = p$, $\dot{c}(0) = X_p$

Mtr : $\nabla_{X_p} Y = \lim_{t \to 0} \frac{1}{t}\left[T(c)_{t \to 0} \, Y_{c(t)} - Y_p \right]$ $(\ast)$

Preuve :

Système (transport $c(\tau) \to c(0)$) $\begin{cases} \frac{d}{dt} \tilde{Y}^i_t + \Gamma^i_{jk} \dot{c}^j \tilde{Y}^k_t = 0 \\ \tilde{Y}(\tau) = Y_{c(\tau)} \end{cases}$

linéarisé en $t = \tau$:

$$\tilde{Y}^i_0(\tau) = Y^i_{c(\tau)} + \Gamma^i_{jk}(c(\tau))\, \dot{c}^j(\tau)\, Y^k_{c(\tau)} (-\tau) + O(\tau^2)$$

$$\frac{1}{\tau}\left[\tilde{Y}^i_0(\tau) - Y^i_p\right] = \frac{Y^i_{c(\tau)} - Y^i_{c(0)}}{\tau} + \Gamma^i_{jk}(c(\tau))\, \dot{c}^j(\tau)\, Y^k_{c(\tau)} + o(\tau)$$

$$\Big\downarrow \tau \to 0$$

$$\nabla_{X_p} Y^i + X_p^j Y_p^k \Gamma^i_{jk}$$

D'où

$$RHS(\ast) = \left(\nabla_{X_p} Y^i\right) e_i + Y^k \nabla_{X_p} e_k = \nabla_{X_p} Y = LHS(\ast)$$

Exo 7 / pg 31 $\quad D^2 f(X,Y) := XYf - (\nabla_X Y) f$, $G := \det(g_{ij})$

1 $\quad D^2 f(X,Y) = D^2 f(Y,X)$

(divergence) 2 $\quad \operatorname{div} X := \operatorname{tr}(Y \longmapsto \nabla_Y X) = \frac{1}{\sqrt{G}} \partial_i (\sqrt{G} X^i)$

(laplacian) 3 $\quad \Delta f := - \operatorname{tr} D^2 f = - \operatorname{div}(\nabla f) = - \frac{1}{\sqrt{G}} \partial_j (g^{jk} \sqrt{G}\, \partial_k f)$

Preuve

① $D^2 f(X,Y) = XYf - YXf + YXf - (\nabla_X Y)f$

$= ([X,Y] - \nabla_X Y)\cdot f + YXf$

$= YXf - (\nabla_Y X)\cdot f = D^2 f(Y,X)$

② • $\nabla_{e_i} X = (\nabla_{e_i} X^j) e_j + X^j \Gamma_{ij}^{\ k} e_k$

$\Rightarrow \mathrm{tr}(Y \longmapsto \nabla_Y X) = \partial_i X^i + X^j \Gamma_{ij}^{\ i}$ (1)

• $\frac{1}{\sqrt{G}} \partial_i (\sqrt{G} X^i) = \partial_i X^i + \frac{X^i}{2G} \partial_i G$

$= \partial_i X^i + \frac{X^i}{2G}\left[G . \mathrm{tr}\left((g_{uv})^{-1}_{uv} \cdot (\partial_i g_{u'v'})_{u'v'}\right)\right]$

$= \partial_i X^i + \frac{X^i}{2} \cdot g^{uv} \underbrace{\partial_i g_{vu}}_{\partial_i \langle e_v, e_u \rangle = \Gamma_{iv}^{\ \ell} g_{\ell u} + \Gamma_{iu}^{\ \ell} g_{v\ell}}$

$= \partial_i X^i + X^i \Gamma_{i\ell}^{\ \ell}$ (2)

(1), (2) $\Rightarrow$ CQFD

③ • $(D^2 f)_{ij} = \partial_i \partial_j f - \Gamma_{ij}^{\ k} \partial_k f$

$\Rightarrow \mathrm{tr}\, D^2 f = g^{ij} \partial_i \partial_j f - g^{ij} \Gamma_{ij}^{\ k} \partial_k f$

• Par (2)

$\mathrm{div}\, \nabla f = \partial_i (g^{ij} \partial_j f) + g^{ij} \partial_j f\, \Gamma_{i\ell}^{\ \ell}$

$= g^{ij} \partial_i \partial_j f + \partial_i g^{ij} \partial_j f + \cancel{g^{ij} \partial_j f\, \Gamma_{i\ell}^{\ \ell}}$

$\partial_i g^{ij} = \nabla_{e_i} \langle dx^i, dx^j \rangle = \cancel{- \Gamma_{ik}^{\ i} g^{kj}} - \Gamma_{ik}^{\ j} g^{ik}$

$\Rightarrow \mathrm{div}\, \nabla f = g^{ij} \partial_i \partial_j f - \Gamma_{ik}^{\ j} \partial_j f \cdot g^{ik}$

D'où $\boxed{\mathrm{tr}\, D^2 f = \mathrm{div}\, \nabla f}$

$\bullet\ \mathrm{div}(\nabla f) = \frac{1}{\sqrt{G}}\, \partial_j \left(\sqrt{G}\, g^{kj} \partial_k f\right)$ $\Rightarrow$ deuxième égalité

Exo 8 / page 31

① $\nabla(fh) = f\nabla h + h\nabla f$ et $\mathrm{div}(fX) = Xf + f\,\mathrm{div}\,X$

② $\mathrm{div}(h\nabla f) = -h\Delta f + \langle \nabla f, \nabla h\rangle$

③ $\Delta(fh) = h\Delta f - 2\langle \nabla f, \nabla h\rangle + f\Delta h$

Preuve

① $\bullet\ \nabla(fh) = g^{ij}\partial_j(hf)\, e_i = g^{ij}(h\partial_j f + f\partial_j h)\, e_i$
$= f\nabla h + h\nabla f$

$\bullet\ \mathrm{div}(fX) = \mathrm{tr}(Y \longmapsto \nabla_Y(fX)) = \mathrm{tr}(Y \longmapsto f\nabla_Y X) + \mathrm{tr}(Y \longmapsto Y(f)X)$
$= f\,\mathrm{div}\,X + X(f)$

② $\mathrm{div}(h\nabla f) = \mathrm{tr}(Y \longmapsto \nabla_Y(h\nabla f))$

$$= \underbrace{\mathrm{tr}(Y \longmapsto \nabla_Y h\, \nabla f)}_{} + \underbrace{h\,\mathrm{tr}(Y \longmapsto \nabla_Y \nabla f)}_{}$$

$= \partial_i h\, g^{ki}\partial_k f \qquad - h\,\mathrm{div}\,\nabla f$

$= \langle \nabla h, \nabla f\rangle - h\Delta f$

③ $\Delta(fh) = -\mathrm{div}(\nabla(fh)) \overset{①}{=} -\mathrm{div}(f\nabla h + h\nabla f)$
$\overset{②}{=} f\Delta h - \langle \nabla f, \nabla h\rangle + h\Delta f - \langle \nabla f, \nabla h\rangle$
$= h\Delta f - 2\langle \nabla f, \nabla h\rangle + f\Delta h$

Exo 9/ pg 31 $\quad \omega$: k-forme

$$\Rightarrow d\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i} \omega)(X_0, \dots, \hat{X_i}, \dots, X_k)$$

Preuve
$$d\omega(X_0, \dots, X_k) = \sum_0^k (-1)^i X_i \cdot \omega(X_0, \dots, \hat{X_i}, \dots, X_k) + \underbrace{\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots)}_{A}$$

$$= \text{RHS} + \sum_{i=0}^{k} (-1)^i \left[\sum_{j \neq i} \omega(\dots, \nabla_{X_i} X_j, \dots) \right] + A$$

$$= \text{RHS} + \sum_{i<j} (-1)^{i+j-1} \omega(\nabla_{X_i} X_j, X_0, \dots, \hat{X_i}, \dots, \hat{X_j}, \dots, X_k)$$
$$+ \sum_{i>j} (-1)^{i+j} \omega(\nabla_{X_i} X_j, X_0, \dots, \hat{X_i}, \dots, \hat{X_j}, \dots, X_k) + A$$

$$= \text{RHS} + \underbrace{\sum_{i<j} (-1)^{i+j} \omega([X_j, X_i], X_0, \dots, \hat{X_i}, \dots, \hat{X_j}, \dots, X_k)}_{-A} + A$$

$$= \text{RHS}$$

Champ de Jacobi & Dérivé de Exp

Def : Champ de Jacobi le long d'une courbe γ :

$$\begin{cases} \dfrac{d^2}{dt^2} Y(t) + R(Y(t), \dot\gamma(t))\dot\gamma(t) = 0 \\ Y(0) = \text{donné} \quad , \quad \dfrac{d}{dt} Y(0) = \text{donné} \end{cases}$$

Prop : $\begin{cases} \gamma : \text{géodésique} \\ \gamma(0) = p \end{cases}$, $\xi \in T_p M$, $Y(t) = t \, D\exp_p(tu) \cdot \xi$
$$= \frac{\partial \alpha}{\partial s}(t, 0)$$
où $\alpha(t,s) = \exp_p t(u + s\xi)$

Alors Y : champ de Jacobi avec $Y(0) = 0$, $\nabla Y(0) = \xi$

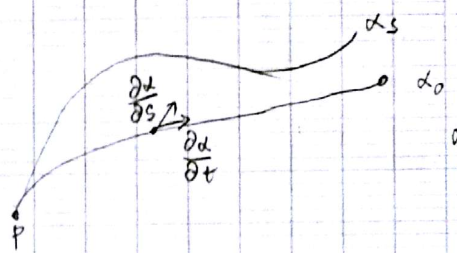
Lemme (Gauß) γ : géodésique, $\gamma(0) = p$, $\dot\gamma(0) = u \in T_pM$

Alors, ① $D\exp_p(tu) \cdot u = \dot\gamma(t)$

② $\langle D\exp_p(tu)\cdot\xi, \dot\gamma(t)\rangle = \langle \xi, u\rangle$

Preuve : Passer à $Y(t)$: ① sol. explicite : $Y(t) = \dot\gamma(t)$

② Vérifier $\dfrac{d^2}{dt^2}\langle Y(t), \dot\gamma(t)\rangle = 0$

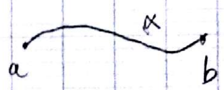


$\alpha(t,s) = \exp_p t(u + s\xi)$

GÉODÉSIQUE

Thm (First Variation Formula)

Contexte :



c : courbe C^∞ par morceaux, $\alpha(s,t) = \exp_{c(t)} sW(t)$

W : champ de variation (défini sur c)

$\dfrac{\partial\alpha}{\partial s} = W$

Alors,

$$DL(c)\,W = \frac{d}{ds} L(\alpha_s)\Big|_{s=0} = -\int_a^b \left\langle W(t), \nabla_{\partial_t} \frac{\dot c}{|\dot c|}\right\rangle dt$$

$$+ \sum_{i=1}^{n-1} \left\langle W(t_i), \frac{\dot c(t_i^-)}{|\dot c(t_i^-)|} - \frac{\dot c(t_i^+)}{|\dot c(t_i^+)|}\right\rangle$$

Remarque $\begin{cases} DL(c).W = 0 \quad \forall W \\ \text{vitesse} = \text{const} \end{cases} \Rightarrow C^\infty$ et géodésique.

Thm (Géodésiques sont minimisantes localement)

Si $\exp_p : B_\varepsilon(O_p) \longrightarrow U \ni p$ ouvert de M est difféo.
Alors $\forall q \in U$
- $\exists!$ géodésique minimisante qui relie p, q
- C'est $\exp_p(tu)$ où $\exp_p au = p$, $|u| = 1$

Preuve : Suffit à montrer $t \longmapsto tu$ est plus courte avec la distance tirée en arrière par $\exp_p$

Soit φ une courbe dans T_pM qui relie O_p et $\exp_p^{-1}(q)$

Par lemme de Gauß (à $\gamma : \tau \longrightarrow \tau . e(t)$ où $\varphi(t) = \underset{\in\mathbb{R}}{r(t)}\ \underset{\text{unitaire}}{e(t)}$)

D'où $\| D\exp_p(\varphi(t)) . e(t) \| = 1$.

Or $D\exp_p(\varphi(t)) . \dot\varphi(t) = \underbrace{D\exp_p(\varphi(t))\, \dot r(t) e(t) + D\exp_p(\varphi(t))\, r(t)\, \dot e(t)}_{\perp \text{ par lemme de Gauß } (\text{à } \xi = \dot e(t),\ \text{même } \gamma, u)}$

$\Rightarrow \| D\exp_p(\varphi(t)) . \dot\varphi(t) \| \geqslant \| \dot r(t) \|$, c-à-d

(Longeur dans M de φ) $\geqslant$ (Longeur dans T_pM) D'où CQFD.

Intépretation géométrique

Def Flot géodésique : flot Φ^t du champ de vecteur suivant dans TM
$\forall u = (p, \xi) \in TM$ ($\xi \in T_pM$), $\exists! S_u \in H_pM$ horizontal
où $T\pi_M S_u = u$

Rmq $\Phi^t_* u = \dot\gamma_u(t)$ où γ_u : géodésique qui commence en p, de vit. ξ.

Thm (Champ de Jacobi = poussé par le flot géodésique)

Contexte : • $u = (p, X) \in TM$, identifier $T_u TM = \underbrace{T_p M}_{H_u} \oplus \underbrace{T_p M}_{V_u}$

• γ_u : géodésique (dans M) commençant par u

• $(A(t), B(t)) = \Phi^t_{u,*} (A, B)$, $(A, B) \in T_u TM$

Alors : $(A(t), B(t)) = (Y(t), \nabla_t Y(t))$ où Y : champ de Jacobi le long γ_u

Preuve : ① Posons • ξ : petite courbe dans TM $\begin{cases} \text{commençant en } u \\ \text{de direction } (A, B) \end{cases}$

• $c(s) = \pi_M \, \xi(s) \rightsquigarrow$ petite courbe commençant en p

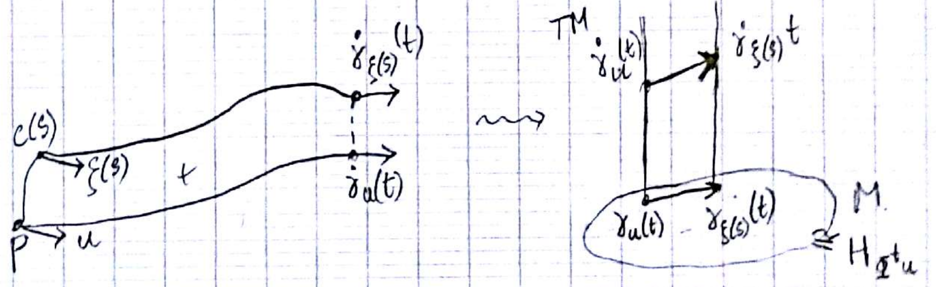
• $\alpha(t, s) = \exp_{c(s)} t\, \xi(s)$

Alors : $Y(t) = \left.\dfrac{\partial \alpha}{\partial s}\right|_{s=0}$ champ de Jacobi le long γ_u

avec $Y(0) = A$, $\nabla_t Y(0) = B$ (calculs direct !)

② $(A(t), B(t)) = \Phi^t_{u,*} (A, B) = \left.\dfrac{d}{ds}\right|_{s=0} \Phi^t(\xi(s)) = \left.\dfrac{d}{ds}\right|_{s=0} \dot{\gamma}_{\xi(s)}(t)$

③



$$A(t) = T_{\Phi^t_p} \pi_M \left(\left.\frac{d}{ds}\right|_{s=0} \dot{\gamma}_{\xi(s)}(t) \right) = \left.\frac{d}{ds}\right|_{s=0} \gamma_{\xi(s)}(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} \alpha = Y(t)$$

$$B(t) = \pi_{V_{\dot{\gamma}_u(t)}} \left(\left.\frac{d}{ds}\right|_{s=0} \dot{\gamma}_{\xi(s)}(t) \right) = \nabla_{\frac{\partial}{\partial s} \gamma_{\xi(s)}(t)} \left.\dot{\gamma}_{\xi(s)}(t)\right|_{s=0}$$

$$= \nabla_{\frac{\partial}{\partial s} \alpha(t,s)} \left.\frac{\partial}{\partial t} \alpha(t,s)\right|_{s=0} = \nabla_{\frac{\partial}{\partial t}} \left.\frac{\partial}{\partial s} \alpha(t,s)\right|_{s=0}$$

$$= \nabla_t Y(t)$$

Def : • <u>Seconde forme fondamentale</u> $N \xrightarrow{i} M$ sous var. riemannienne

$$S(X,Y) := \left(\nabla_X^M Y\right)^{\perp} \longrightarrow \text{projection on } TN$$

$TM = TN \oplus TN^{\perp}$
$X, Y \in \Gamma(TN)$

• <u>Shape operator</u> $\xi \in T_pN^{\perp}$, $A_\xi : T_pN \longrightarrow T_pN$

$$\langle A_\xi X, Y\rangle = -\langle S(X,Y), \xi\rangle$$

<u>Remarque</u> : • S : tenseur symétrique

• A_ξ : symétrique $\langle A_\xi X, Y\rangle = \langle X, A_\xi Y\rangle$

$A_\xi X = \left(\nabla_X^M \tilde{\xi}\right)^T$ for any extension $\tilde{\xi}$ of ξ on N

• $\left(\nabla_X^M Y\right)^T = \nabla_X^N Y \quad \forall X, Y \in TN$

Exo 6 / pg 48

① (Gauß) $x, y, z, w \in T_pN$

a) $\left[R^M(x,y)z\right]^T = R^N(x,y)z + A_{S(y,z)}x - A_{S(x,z)}y$

b) $R^M(x,y,z,w) = R^N(x,y,z,w) + \langle S(x,z), S(y,w)\rangle$
$\quad - \langle S(y,z), S(x,w)\rangle$

c) $K^N(x,y) = K^M(x,y) + \langle S(x,x), S(y,y)\rangle$
$\quad - \langle S(x,y), S(x,y)\rangle$

② (Weingarten) $\begin{cases} \xi \text{ champ } \mathcal{C}^\infty \text{ défini sur } N, \text{ normal} \\ X \in \Gamma(TN) \\ \nabla_X^{\perp}\xi = \left(\nabla_X^M \xi\right)^{\perp} \end{cases}$

$\nabla^{\perp}$ connection sur ν_N et $\nabla_X^M \xi = A_\xi X + \nabla_X^{\perp}\xi$

③ (Codazzi)

$$\left[R^M(X,Y)Z\right]^{\perp} = (\tilde{\nabla}_X S)(Y,Z) - \tilde{\nabla}_Y S(X,Z)$$

où $\tilde{\nabla}_X S(Y,Z) := \nabla_X^{\perp} S(Y,Z) - S(\nabla_X^N Y, Z) - S(Y, \nabla_X^N Z)$

Exo (Cheeger-Gromoll) Soit N le gradient d'une fonction b sur une variété riemannienne M. Soit $\{E_i\}_{i=1}^{n-1}$ et N une frame local autour d'un point x qui soit parallel le long de la courbe integrale passant par x. Alors si $\|N\| = 1$

$$Ric(N,N) = N(\Delta b) - \|\nabla N\|^2$$

Preuve :

$$Ric(N,N) = \sum_{i=1}^{N-1} \langle R(E_i, N)N, E_i \rangle$$

$$= \sum_{i=1}^{n-1} \langle \nabla_{E_i} \nabla_N N - \nabla_N \nabla_{E_i} N - \nabla_{[E_i,N]} N, E_i \rangle$$

$$= \sum_{i=1}^{n-1} - \langle \nabla_N \nabla_{E_i} N, E_i \rangle - \langle \nabla_{\nabla_{E_i} N} N, E_i \rangle$$

$$= \sum_{i=1}^{n-1} - N \langle \nabla_{E_i} N, E_i \rangle$$

$$- \sum_{1 \leq i,j \leq n-1} \langle \nabla_{E_i} N, E_j \rangle \langle \nabla_{E_j} N, E_i \rangle$$

$$= \sum_{i=1}^{n-1} N \langle N, \nabla_{E_i} E_i \rangle - \|\nabla N\|^2$$

$$= N(\Delta b) - \|\nabla N\|^2$$

Exo (Wikipedia + Demailly Chap VIII, §11)

Etant donné un tenseur A de type $(1,1)$, le <u>tenseur de Nijenhuis</u> de A est définie par

$$N_A(X,Y) = -A^2[X,Y] + A([AX,Y]+[X,AY]) - [AX,AY]$$

Mtrer que $N_J = 0$ si J est une structure presque complexe parallèle

<u>Preuve</u> Comme le connection de levi-Civita est sans torsion :

$$N_J(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY]$$

$$= \nabla_X Y - \nabla_Y X + J(\nabla_{JX} Y - \nabla_Y JX) + J(\nabla_X JY - \nabla_{JY} X) - (\nabla_{JX} JY - \nabla_{JY} JX)$$

(Car $\nabla_A JB = J\nabla_A B$, J est parallèle)

$$= \nabla_X Y - \nabla_Y X + J\nabla_{JX} Y \cancel{- J\nabla_Y JX} + \cancel{J\nabla_X JY} - J\nabla_{JY} X \cancel{- J\nabla_X JY} + \cancel{J\nabla_Y JX}$$

$$= \nabla_X Y - \nabla_Y X + J\nabla_{JX} Y - J\nabla_{JY} Y$$

Donc $$(N_J)^i_{jk} = -J^h_j \partial_h J^i_k + J^h_k \partial_h J^i_j + J^i_h \partial_j J^h_k - J^i_h \partial_k J^h_j \quad (*)$$

Soit $J = J^j_i \, e_j \otimes dx^i$ en coordonnés locales, alors

$$0 = \nabla_{e_\ell} J = (\partial_\ell J^j_i + J^k_i \Gamma^j_{k\ell} - J^j_k \Gamma^k_{i\ell}) \, e_j \otimes dx^i$$

Donc, $\partial_\ell J^j_i = J^j_k \Gamma^k_{i\ell} - J^k_i \Gamma^j_{\ell k}$. Substituer dans $(*)$

$$(N_J)^i_{jk} = -J^h_j (J^i_\ell \Gamma^\ell_{kh} - J^\ell_k \Gamma^i_{\ell h}) + J^h_k (J^i_\ell \Gamma^\ell_{hj} - J^\ell_j \Gamma^i_{h\ell}) + J^i_h (J^h_\ell \Gamma^\ell_{kj} - J^\ell_k \Gamma^h_{j\ell}) - J^i_h (J^h_\ell \Gamma^\ell_{kj} - J^\ell_j \Gamma^h_{k\ell})$$

$$= 0$$

Donc $N_J = 0$ et J est intégrable par Thm de Newlander-Nirenberg.

Exo (Prop. 4.A 6 - 4.A.7, pg 208-209, Huybrechts) X : $\begin{cases} \text{almost complex} \\ \text{Riemannienne} \end{cases}$

① $\nabla : T^{1,0}$ hermitienne $\Longrightarrow$ $D : TX$ connexion, $Dg = 0$

② $\begin{cases} \nabla \text{ hermitienne} \\ T_\nabla := T_D = 0 \end{cases} \Rightarrow \begin{cases} \nabla : \text{Chern.} \\ D : \text{levi-Civita} \\ X : \text{Kähler} \end{cases}$

Preuve $\quad \xi : \underset{\frac{1}{2}(-,-)}{TX} \longrightarrow \underset{\langle -,- \rangle_{\mathbb{C}}}{T^{1,0}X}$

① Posons $D_v u := \xi^{-1}(\nabla(\xi(u)).v)$. Soit $\begin{cases} s = \xi u \\ t = \xi v \\ w \in T(TX) \end{cases}$

- $d\{s,t\}.w = D_w\left(\frac{1}{2}(\langle u,v \rangle - i\,\omega(u,v))\right)$ (1)

- $\{\nabla s, t\}.w + \{s, \nabla t\}.w = \{\nabla s.w, t\} + \{s, \nabla t.w\}$

$= \langle \xi(D_w u), \xi v \rangle_{\mathbb{C}} + \langle \xi u, \xi D_w v \rangle_{\mathbb{C}}$

$= \frac{1}{2}(\langle D_w u, v \rangle - i\,\omega(D_w u, v)) + \frac{1}{2}(\langle u, D_w v \rangle - i\,\omega(u, D_w v))$ (2)

Prenons la partie réelle de (1), (2) :

$$D_w \langle u,v \rangle = \langle D_w u, v \rangle + \langle u, D_w v \rangle \Rightarrow Dg = 0$$

②. D est levi-Civita par déf.

- Suffit à mtrer $D\omega = 0$ car $d\alpha = \sum D_{v_i} \alpha(v_0, \ldots, \hat{v}_i, \ldots)$

qui vient de la partie imaginaire de (1), (2)

- $T_\nabla = 0 \Rightarrow \nabla = d + A_\nabla$ où $A_\nabla(u).\xi(v) = A_\nabla(v).\xi(w)$

Suffit à mtrer que $A_\nabla \in \mathcal{A}^{1,0}(\mathrm{End}(T^{1,0}X))$, c-à-d

$$\text{c-à-d}\quad \tilde{A}_\nabla(\underbrace{u + iJu}_{\in\, T^{0,1}X}) = 0 \quad (\tilde{A}_\nabla : \text{extension of } A_\nabla \text{ to } T_{\mathbb{C}}X)$$

$$\text{Or,}\quad [A_\nabla(u) + i A_\nabla(Ju)] \cdot \xi(v) = A_\nabla(v) \cdot \xi(u) + i A_\nabla(v)(\xi(Ju))$$

$$= A_\nabla(v) \cdot \xi(u) + i A_\nabla(v)(i\,\xi(u)) = 0 \quad (\text{CQFD})$$

Exo (Prop 4.A.8, pg 210, Huybrechts) $\left\{ \begin{array}{l} (X,g) \text{ Riemannienne.} \\ J : \text{structure presque complexe parallèlle} \end{array} \right.$

① D : TX levi-Civita donne.
∇ : $T^{1,0}X$ hermitienne.

② X est kahlerienne, $\nabla\omega = 0$

<u>Preuve</u> ① $(\nabla s) \cdot v := \xi(D_v\, \xi^{-1}(s))$ satisfait

- $\nabla(fs) \cdot v = v(f)\, s + f\, \nabla s \cdot v$ $\Big\}$ Automatique
- $\mathbb{R}$-linéaire en v
- $\mathbb{C}$-linéaire en s car $\xi D_v(\xi^{-1}(is)) = \xi\, D_v(J\xi^{-1}(s)) = \xi(J D_v\, \xi^{-1}(s))$
$$= i\,\xi D_v\, \xi^{-1}(s)$$

<u>Or</u> . $\nabla \langle s,t\rangle_{\mathbb{C}} \cdot w = D_w\left(\frac{1}{2}\langle u,v\rangle - \frac{1}{2} i\,\omega(u,v)\right)$ $\quad s = \xi(u)$, $t = \xi(v)$, $w \in \Gamma(TM)$

. $\langle \nabla s, t\rangle_{\mathbb{C}} \cdot w = \langle \xi D_w u, \xi v\rangle_{\mathbb{C}}$
$$= \frac{1}{2}\left(\langle D_w u, v\rangle - \frac{1}{2} i\,\omega(D_w u, v)\right)$$

. $\langle t, \nabla s\rangle_{\mathbb{C}} \cdot w = \langle \xi v, \xi D_w u\rangle_{\mathbb{C}}$
$$= \frac{1}{2}\left(\langle v, D_w u\rangle - \frac{1}{2} i\,\omega(v, D_w u)\right)$$

les parties réelles s'égalent : D : levi-Civita
les parties imaginaires : $D_w(\omega(u,v)) = D_w\langle Ju, v\rangle$
$$= \langle D_w(Ju), v\rangle + \langle Ju, D_w v\rangle = \omega(D_w u, v) + \omega(u, D_w v)$$
$\Rightarrow \nabla\omega = 0$ aussi

Exo 4.A.2 pg 216, Huybrechts

Compléter la preuve de 4.A.9

Preuve : Reste à voir que si

$$\frac{\partial h_{iJ}}{\partial z_k} = \frac{\partial_{kJ}}{\partial z_i}$$

Alors $A = (h_{Ji})^{-1} \partial h_{Ji}$ satisfait $A(u)\cdot v = A(v)\cdot u$ quelque soit $u, v \in T_pM$.

OPS que $u = e_k$, $v = e_\ell$

$$A(e_k)\cdot e_\ell = \left((h_{Ji})^{-1} \frac{\partial h_{Ji}}{\partial e_k} \right)_{i,J} \wedge e_\ell$$

$$= \sum_i (h_{\ell i})^{-1} \frac{\partial h_{\ell i}}{\partial e_k} = \sum_i (h_{ki})^{-1} \frac{\partial h_{ki}}{\partial e_\ell} = A(e_\ell)\cdot e_k$$

Exo 4.A.3 pg 216, Huybrechts (X,g) Kahlerienne, alors

$i\,\rho(X,g) = \Phi_{K_X}$ courbure de la connexion de Cherne
où $\rho(a,b) = Ric(Ja,b)$ est la forme de Ricci

Preuve : La courbure $\Phi_{K_X} = \Phi_{det(\Omega_X)}$ vaut

$$\Phi_{K_X} = tr_{Hom(\Omega_X,\Omega_X)}(\Phi_{\Omega_X})$$

$$= tr_{Hom(\Omega_X,\Omega_X)}(-\Phi_{\tau_X}^{+})$$

$$= -tr_{Hom(\tau_X,\tau_X)}(\Phi_{\tau_X})$$

Or par Proposition 4.A.11 (pg 211) :

$$\rho(X,g) = i\,tr_{Hom(\tau_X,\tau_X)}(\Phi_{\tau_X})$$

(<u>Note</u> : différente notation avec Huybrechts :

$$\rho(X,g) \longleftrightarrow \mathrm{Ric}(X,g)$$

$$\Phi_{\mathcal{L}_X} \longleftrightarrow F_\nabla \quad)$$

On obtient $\quad i\,\rho(X,g) = \Phi_{K_X}$

Exo 1.2.5 pg 40, Huybrechts

- Mtrer que V est un e.v de coeff quaternion.

Suffit à voir v, Iv, Jv, Kv sont perpendiculaires. Or,

$$\langle Iv, Jv \rangle = \langle JIv, J^2 v \rangle = \langle Kv, v \rangle = 0$$

- Mtrer que $\omega_J + i\omega_K$ est une $(2,0)$-forme par rapport à I

Rmq : le fait que $(\omega_J + i\omega_K) \circ I = -(\omega_J + i\omega_K)$ ne nous permet pas à distinguer une forme $(2,0)$ et $(0,2)$.

① $\omega = (\omega_J + i\omega_K)$ satisfait $\omega \circ I = -\omega$:

$$\omega(Iu, Iv) = \langle JIu, Iv \rangle + i \langle KIu, Iv \rangle = -(\omega_J + i\omega_K)(u,v)$$
$$= -\omega(Iu, Iv) \Rightarrow \Pi^{1,1}\omega = 0$$

② ω satisfait $\omega \circ (I, Id) = i\,\omega$:

$$\omega(Iu, v) = \langle JIu, v \rangle + i \langle KIu, v \rangle = -\langle Ku, v \rangle + i \langle Ju, v \rangle$$
$$= i(\omega_J + i\omega_K)(u,v)$$
$$\Rightarrow \omega \in \Lambda^{2,0} V^*$$